LECTURES ON THE THEORY OF FUNCTIONS

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LECTURES ON THE THEORY OF FUNCTIONS

BY

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TO

W. W. ROGOSINSKI

PREFACE

The Introduction and Chapter I were printed off in 1931 and some important changes should be supplied from the Addenda and Corrigenda, which an intending reader should take note of at once. Chapter II, whose completion has been unavoidably delayed, has now been rewritten. For help in this I owe an overwhelming debt to Dr. W. W. Rogosinski, who not only supplied much of the material, but criticised and corrected my text in the last detail.

I wish also to express my gratitude to the printers Messrs. C. F. Hodgson & Son for their courtesy and great forbearance over a difficult 20 years.

June, 1944. J. E. L.

CONTENTS

NOTE

The theorems of the Introduction have numbers below 100, those of Chapter I begin at 101, those of Chapter II at 201. Sections are numbered consecutively.

ERRATUM

p. 153. (i) The upper limit in (A*) is, as in (A)(1), with respect to $\zeta \neq z$.

(ii) The usual definition of upper-semi-continuity is (A)(1) and not (as in the text) (A*). This makes, of course, only a momentary difference.

Introduction.

THE various matters collected in the Introduction agree only in being more conveniently separated from their applications. It is not, however, necessary to read it consecutively, and much of it is first required in Volume 2; the reader may therefore welcome a few words of explanation and advice.

He cannot become familiar too early with the inequalities of Hölder and Minkowski, and he should read consecutively (but not try to memorize) to the end of Section 2 if he can do so without becoming impatient. This section is developed rather more systematically than is necessary for applications, but the number of distinct forms in Theorems 1 and 2 that are specifically used is surprisingly large; and if the details are taken with a judicious lightness the subject is quite an easy one. Section 3 is very short. There is a certain field of complex function theory (the problems of "boundary-values"—these are discussed in Volume 2) which demands a fairly complete "real-variable" technique. Sections 4 and 5 are designed to meet this need, Section 4 dealing with general theory, and Section 5 with the more special subject of Fourier series. While not exhaustive, the account is sufficiently systematic to be read for its own sake, but the reader may postpone it if he wishes until he reaches Volume 2. Section 6 is concerned with an isolated problem of *analysis situs*, and may be read when it becomes relevant (in Section 19). Section 7 presents a fairly complete general theory of harmonic functions; much of it is required later, it is easy, and the subject is apt to be neglected in England; it should probably be read before Chapter I. Section 8 consists of straightforward calculations. It sets out the behaviour of certain special functions whose rôle is to be illustrative, and especially to provide "Gegenbeispiel"'s†. It is required hardly at all in Volume 1.

† A "Gegenbeispiel" for a proposition p is an example which shows that p is false: the function x^{-1} is a "Gegenbeispiel" for the proposition "all functions are bounded in $0<x<1$". The important examples are those which complete the account of a theorem by showing that it is "best possible" (depends on the minimum hypotheses).

1. *Notation.* We use the symbol $A(x, y, \ldots)$, or sometimes $A_{x, y, \ldots}$ for a positive constant depending only on the parameters shown explicitly; in particular A will denote a positive absolute constant. We use K for a positive constant depending in general on all the parameters of the context. We use ϑ for a number satisfying $|\vartheta| \leqslant 1$. A's, K's, and ϑ's are not in general the same from one occurrence to another; if we wish to preserve their identity in the course of an argument we affect them with suffixes 1, 2, ...

$\epsilon(x)$, ϵ_n, etc., denote functions tending to 0 as their argument tends to the limit (finite or infinite) under consideration. The symbol $o(1)$ is available for such functions, and the ϵ notation is used only to mark a distinction; we use it for functions that are independent of some parameter or parameters.

The symbol ϵ without an argument, and also δ, denote as usual positive constants ("arbitrarily small").

Certain letters *used as indices* (exponents) will denote numbers subject to special conditions. μ may be any real constant, positive or negative. The remaining letters denote *positive* constants, and, moreover, are restricted by the following inequalities:

$$\lambda > 0,\ k \geqslant 1,\ r > 1;\quad 1 < p \leqslant 2,\ q \geqslant 2;\quad 0 < \kappa \leqslant 1,\ 0 < \rho < 1.$$

We shall occasionally allow ourselves the licence of extending the ranges of λ, κ, ρ to include 0, those of μ, λ, k, r, q to include $+\infty$, and that of μ to include $-\infty$; but in such cases we shall always indicate the extension explicitly. [The commonest indices are λ and r. p and q do not occur in Vol. 1. The definitions are repeated from time to time, and the reader need not memorize them.]

We write $t' = t/(t-1)$, where t is any one of the special indices (supposed, however, not to have the value $t = 1$). A dashed letter does not necessarily belong to the class denoted by the undashed letter: thus p' and q' are respectively of types q and p, and λ', κ', ρ' may be negative.

The relation between t and t' may be expressed in two further ways, with which the reader should make himself familiar:

$$\frac{1}{t} + \frac{1}{t'} = 1, \quad (t-1)(t'-1) = 1.$$

The integrals with which we shall be concerned are generally extended over a bounded set of points. Such a set of points can be reduced by a trivial transformation to lie within any given interval: we shall suppose always, unless the contrary is stated (and this does sometimes happen), that all sets $E, e, \ldots$ are contained in the interval $(-\pi, \pi)$,

which we denote by E_0. We write $E_1 \subset E_2$ for "E_1 is contained in E_2", and denote $E_0 - E$ by CE.

By HK, the "product" of two sets of points H and K, we mean the set of points common to H and K.

A function $f(\theta)$ to be considered in E_0 is likely to have some natural relation to the period 2π. On balance it pays to lay down the convention that "f is continuous in E_0" shall include the relation $f(-\pi) = f(\pi)$. In theorems about functions not necessarily continuous it is generally possible to alter arbitrarily the value of the function at a single point. In such circumstances we shall tacitly suppose that $f(-\pi) = f(\pi)$ and that f, defined originally in E_0, exists everywhere and has the period 2π. This convention enables us, for example, to treat an interval $|\theta - \theta_0| \leqslant k$ on the same footing when it projects out of E_0 as when it does not.

Unless the contrary is stated all *given* functions are supposed measurable: other questions of measurability are generally trivial, and we do not discuss them.

When $|f(\theta)|^\lambda$ is integrable in the sense of Lebesgue in a set E we say that f belongs to the class L^λ in E. We write also for brevity L in place of L^1.

The "sign of z", or, in symbols, $\operatorname{sgn} z$; is defined to be 0 if $z = 0$ and $z/|z|$ otherwise. $\bar{z}$ denotes the conjugate of z, $\overline{\operatorname{sgn}}\, z = \operatorname{sgn} \bar{z}$.

The symbol $[f]_N$ denotes f if $|f| \leqslant N$, and $N \operatorname{sgn} f$ if $|f| > N$. $[E]_N$ denotes that part of the set E for which the modulus of the variable does not exceed N.

By a null-set we understand a set of zero measure, by a null-function a function that is zero except in a null-set. $f \equiv \phi$, or, in words, "f is equivalent to ϕ", means that $f = \phi$ except in a null-set, or that $f - \phi$ is a null-function.

We shall use the following abbreviations :—

p.p. ("presque partout") for "almost everywhere" or "almost always" (*i.e.* "except in a null-set"). ["a.e." is insufficiently vivid and is apt to be mistaken for other things];

b.v. for "bounded variation" and "of bounded variation";

a.c. for "absolute continuity" and "absolutely continuous";

u.b.v. and u.a.c. for "uniform(ly) b.v." and "uniform(ly) a.c.";

t.v. for "total variation".

By a "trigonometrical polynomial" we understand a finite sum of type

$$\sum_{n=0}^{N} (c_n \cos n\theta + d_n \sin n\theta).$$

2. *The inequalities of Hölder and Minkowski.*

2.1. We suppose until further notice, unless the contrary is stated, that all letters denote numbers that are positive or zero. The sums with which we deal are in general taken over an infinity of terms, but in our *proofs* we may suppose them finite, and complete the argument by a trivial passage to the limit. There is a single exception to this rule: Theorem 4 of § 2.82. Here "convergence" is mentioned explicitly and given a special treatment.

Hölder's inequality is

$$(H) \qquad \Sigma ab \leqslant (\Sigma a^r)^{1/r} (\Sigma b^{r'})^{1/r'} \quad (r > 1).$$

Minkowski's inequality is

$$(M) \qquad \left(\Sigma (a+b)^k\right)^{1/k} \leqslant (\Sigma a^k)^{1/k} + (\Sigma b^k)^{1/k} \quad (k \geqslant 1).$$

We first prove these results, then develop them at length, and finally collect everything for reference in Theorems 1 and 2.

2.2. Let $U^r = \Sigma a^r$, $V^r = \Sigma b^{r'}$, $W = \Sigma ab$. We have

$$(1) \qquad ab \leqslant \frac{a^r}{r} + \frac{b^{r'}}{r'}.$$

For
$$\left(\frac{a^r}{r'} + \frac{b^{r'}}{r}\right) \Big/ ab = t(x) = \frac{x^r}{r} + \frac{x^{-r'}}{r'},$$

where
$$x = a^{1/r'} b^{-1/r}.$$

and differentiation shows that $t(x)$ is a minimum (for $x \geqslant 0$) when $x = 1$, in which case $t = 1$.

It follows from (1) that if λ is any positive constant

$$ab = \lambda a . \lambda^{-1} b \leqslant \lambda^r \frac{a^r}{r} + \lambda^{-r'} \frac{b^{r'}}{r'}.$$

Summing we have

$$(2) \qquad W \leqslant \lambda^r \frac{U^r}{r} + \lambda^{-r'} \frac{V^r}{r'}.$$

We may suppose in (H) that U, $V > 0$, in which case, if we choose λ so that

$$\lambda^r U^r = \lambda^{-r'} V^{r'} = (\lambda^r U^r)^{1/r} (\lambda^{-r'} V^{r'})^{1/r'} = UV,$$

(2) becomes

$$W \leqslant \frac{UV}{r} + \frac{UV}{r'} = UV,$$

and this is (H).

The inequality (M) is trivial when $k=1$; supposing then $k>1$ we have, by (H),

$$
\begin{aligned}
T^k &= \Sigma(a+b)^k = \Sigma(a+b)^{k-1}a+\Sigma(a+b)^{k-1}b \\
&\leqslant \{\Sigma(a+b)^k\}^{1/k'}(\Sigma a^k)^{1/k}+\{\Sigma(a+b)^k\}^{1/k'}(\Sigma b^k)^{1/k} \qquad (3)\\
&= T^{k-1}\{(\Sigma a^k)^{1/k}+(\Sigma b^k)^{1/k}\},
\end{aligned}
$$

and the desired result follows.

(M) evidently extends directly (or by induction) to more than two sets of numbers (a), (b); we have, in fact,

$$
\{\Sigma(a+b+c+\ldots)^k\}^{1/k} \leqslant (\Sigma a^k)^{1/k}+(\Sigma b^k)^{1/k}+\ldots. \qquad (4)
$$

Results corresponding to (H) and (M) exist also with integrals in place of sums, and in (4), where a double summation is involved, there are also mixed forms. For the most part the proofs are substantially the same for sums or integrals; where this is so we shall generally give only the *argument* for sums; where it is not the integral case is the more difficult and we consequently select it. In *stating* results we select sometimes the sum, sometimes the integral form. We suppose in our proofs that the range of integration is bounded; extensions to infinite range are trivial when they are valid, and we do not consider them until our final summing up. Our integrals are Lebesgue integrals. We actually require none but elementary integrals in Volume 1, but the subject is more *easily* treated in the general field, and the full results are, in any case, required in Volume 2.

For the "integral-integral" form of (4) the argument transforms as follows: The case $k=1$ is trivial. Supposing then $k>1$ we have

$$
T^k = \int dy\left(\int f(x,y)\,dx\right)^k = \int dx\left\{\int f\left(\int f dx\right)^{k-1}dy\right\}
$$

$$
\leqslant \int dx\left\{\left[\int f^k dy\right]^{1/k}\left[\int\left(\int f dx\right)^k dy\right]^{1/k'}\right\} = \int dx\left[\int f^k dy\right]^{1/k} T^{k/k'}, \qquad (5)
$$

or

$$
T \leqslant \int dx\left(\int f^k dy\right)^{1/k},
$$

which is the desired result.

2.3. Let now f and g be functions, possibly complex, for which $g \not\equiv 0$. Then

$$
\frac{1}{2\pi}\left|\int_{-\pi}^{\pi} fg\,d\theta\right| \leqslant \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^r d\theta\right)^{1/r}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|g|^{r'}d\theta\right)^{1/r'} = M_r(f)M_{r'}(g). \qquad (1)
$$

This is, in fact, what may be called the "mean" form of (H) (for

integrals). The 2π's may be retained or omitted at our pleasure, since they occur to the same power -1 on both sides. We prove now that the sign of equality in (1) holds† if and only if each of

(2) $$|f|^r \equiv c|g|^{r'}, \quad \text{where} \quad c = M_r^r(f)/M_{r'}^{r'}(g),$$

and

(3) $$\operatorname{sgn} fg \equiv e^{i\alpha} \equiv c\,|g|^{r'}, \quad \text{where } \alpha \text{ is a real constant,}$$

hold in the set of θ for which $f \neq 0$.

It is easily seen that equality does hold in (1) subject to (2) and (3). Suppose now that equality holds. Then, in the first place, it continues to hold when the integrand fg is replaced by $|f|\,|g|$. Let $\lambda^{r+r'} = c^{-1}$. Then (indeed for any λ)

(4) $$|fg| \leqslant \lambda^r \frac{|f|^r}{r} + \lambda^{-r'} \frac{|g|^{r'}}{r'},$$

and equality in (4) happens only if $|f|^r = c\,|g|^{r'}$. If (2) is false there exists a set, not null, in which (4) holds with inequality, and therefore a non-null set in which the difference of the two sides exceeds some positive δ‡. Then

$$\int_e |fg|\,d\theta < \frac{\lambda^r}{r}\int_e |f|^r\,d\theta + \frac{\lambda^{-r'}}{r'}\int_e |g|^{r'}\,d\theta.$$

Since in any case

$$\int_{E_0-e} |fg|\,d\theta \leqslant \frac{\lambda^r}{r}\int_{E_0-e} |f|^r\,d\theta + \frac{\lambda^{-r'}}{r'}\int_{E_0-e} |g|^{r'}\,d\theta,$$

† The reader will find in Vol. 2 that the conditions for equality (here and in §2.76) can be important weapons of argument: it is a mistake to suppose that they are of purely academic interest.

‡ We shall often have to use the principle involved here, which is that if $\phi(\theta) > 0$ in a set of positive measure, then, for some δ, $\phi > \delta$ in a set of positive measure. The principle can be generalized into the following form.

Suppose that with every θ of a set E of positive measure there are associated h positive numbers $\phi_1(\theta), \phi_2(\theta), \ldots, \phi_h(\theta)$; k finite real numbers $M_1(\theta), M_2(\theta), \ldots, M_k(\theta)$; and l positive integers $N_1(\theta), N_2(\theta), \ldots, N_l(\theta)$. Then there exists a positive number a, a finite μ, l positive integers $\nu_1, \nu_2, \ldots, \nu_l$, all independent of θ, and a perfect set E^* of positive measure contained in E, such that for every θ of E^*

$$\phi_n(\theta) > a \quad (n \leqslant h), \qquad |M_n(\theta)| < \mu \quad (n \leqslant k), \qquad N_n = \nu_n \quad (n \leqslant l).$$

In fact, let $H(p, q; r_1, r_2, \ldots, r_l)$ be the set of θ of E for which $\phi_n(\theta) > p^{-1}$ $(n \leqslant h)$, $|M_n(\theta)| < q$ $(n \leqslant k)$, $N_n = \nu_n$ $(n \leqslant l)$. Every θ of E belongs to some set H, and $E = \Sigma H$, the summation being taken over all positive integral $p, q, r_1, \ldots, r$. Since Σ has a denumerable number of terms, some H has positive measure with E, since $mE \leqslant \Sigma mH$. H contains a perfect set E^* of positive measure, and this satisfies the required conditions, with

$$a = p^{-1}, \quad \mu = q, \quad \nu_n = r_n.$$

we have by addition

$$\frac{1}{2\pi}\int_{E_0}|fg|\,d\theta < \frac{\lambda^r}{r}\frac{1}{2\pi}\int_{E_0}|f|^r d\theta + \frac{\lambda^{-r'}}{r'}\frac{1}{2\pi}\int_{E_0}|g|^{r'} d\theta$$

$$= \left(\frac{1}{r}+\frac{1}{r'}\right) M_r(f)\, M_{r'}(g) = M_r(f)\, M_{r'}(g),$$

contrary to hypothesis. Thus (2) must hold.

Finally, for equality in (1) we must have

$$\int_{E_0} fg\,d\theta = e^{i\beta}\int_{E_0}|fg|\,d\theta$$

$$\int_{E_0}|fg|(1-e^{-i\beta}\operatorname{sgn} fg)\,d\theta = 0.$$

The real part of the integrand being non-negative, we must have

$$|fg|\,\{1-\Re(e^{-i\beta}\operatorname{sgn} fg)\} \equiv 0.$$

Since the set in which $fg = 0$ is equivalent, by (2), to the set in which $f = 0$, we have $\Re(e^{-i\beta}\operatorname{sgn} fg) \equiv 1$ and so $e^{-i\beta}\operatorname{sgn} fg \equiv 1$, except when $f = 0$. This completes the proof.

The case of sums is much simpler.

Consider now the case of equality in the (M) inequalities, supposing everything non-negative. There is equality in all cases if $k = 1$. If $k > 1$ the condition for equality in the "integral-integral" form is that $f(x, y) = F(x)\, G(y)$ p.p. in x and p.p. in y. In fact, for equality in (5) of § 2 . 2 the x-integrands must be equal p.p. in x. By the (H) result equality requires

$$\frac{\{f(x, y)\}^k}{\left\{\left(\int f\,dx\right)^{k-1}\right\}^{k'}} = c(x),$$

p.p. in y, where $c(x)$ is independent of y. This proves the result.

In the "sum-sum" form (4) the condition is that $b_n = ca_n$, $c_n = c'a_n$, ... for all n, where c, c', ... are positive constants.

2 . 4. The inequality (H) remains valid if the index r is replaced by a $\mu < 1$ and the sign of inequality is reversed, provided only that $\mu \neq 0$ [negative values of μ are permitted]. Similarly (M) is true if k is replaced by $\mu \leqslant 1$ and the sign of inequality is reversed, provided $\mu \neq 0$.

To prove this let us denote the inequality (H) by $I(a, b, r)$, and the inequality with reversed sign by I^*. If now $\mu = -\lambda < 0$ and we write $a = \alpha^{-(\lambda+1)/\lambda}$, $b = (\alpha\beta)^{(\lambda+1)/\lambda}$, $I^*(a, b, -\lambda)$ is equivalent to $I(\alpha, \beta, 1+\lambda^{-1})$, which is true. If $\mu = \rho$ $[0 < \rho < 1]$ and we write $a = (\alpha\beta)^{1/\rho}$, $b = \beta^{-1/\rho}$,

then $I^*(\alpha, b, \rho)$ is equivalent to $I(\alpha, \beta, 1/\rho)$, which is true. Thus our assertion about (H) is proved. In the case of (M) we have only to carry out our former proofs, using I^* in place of I.

2.5. LEMMA α $$(\Sigma a) \geqslant (\Sigma a^k)^{1/k} \quad (k \geqslant 1).$$

For $$(\Sigma a)^k = \Sigma\{(\Sigma a)^{k-1}.\, a\} \geqslant \Sigma\{a^{k-1}.\, a\}.$$

[The simplest case of the lemma is
$$(1+x)^k \geqslant 1+x^k \quad (x \geqslant 0).]$$

The inequality (H) extends at once to the form

(1) $$|\Sigma ab \ldots| \leqslant (\Sigma|a|^{r_1})^{1/r_1}(\Sigma|b|^{r_2})^{1/r_2}\ldots,$$

where the r's are connected by

(2) $$\Sigma\frac{1}{r} = 1,$$

and the a's, b's, ... are not necessarily positive. To prove this we write the product $ab \ldots$ as $a\beta$ and use (H) with $r = r_1$. In the sum $\Sigma\beta^{r_1'}$ we now write β as $b\gamma$ and use (H) with $r = r_2$, and so on. We thus obtain (1).

We observe next that (1) remains true subject only to

(3) $$r_1 > 1,\ r_2 > 1,\ \ldots,\quad \Sigma\frac{1}{r} \geqslant 1$$

In fact, let $\Sigma 1/r = k$, or $\Sigma 1(rk) = 1$. Then, by Lemma α,

$$\Sigma ab \ldots t| \leqslant (\Sigma|a|^{1/k}\ldots|t|^{1/k})^k \leqslant [\Pi\{\Sigma(|a|^{1/k})^{rk}\}^{1/(rk)}]^k = \Pi(\Sigma|a|^r)^{1/r}.$$

The inequality (1), *subject to* (2), may be replaced by the "mean" form
$$\left|\frac{1}{n}\Sigma ab\ldots\right| \leqslant \Pi\left(\frac{1}{n}\Sigma|a|^r\right)^{1/r},$$

in which n is the number of terms in each set of numbers (a), (b), ... *This* inequality does not hold subject to (3). We shall call inequalities "homogeneous"† when they are true equally in "sum (integral)" or "mean" form.

We conclude this paragraph by noting some easy variants and consequences of (H). The integrals are all taken over $(-\pi, \pi)$, and f, g, ... are not necessarily positive.

(4) $$\left|\frac{1}{2\pi}\int fg\,d\theta\right|^k \leqslant \left(\frac{1}{2\pi}\int|f^k g|\,d\theta\right)\left(\frac{1}{2\pi}\int|g|\,d\theta\right)^{k-1} \quad (k \geqslant 1).$$

† The homogeneity is in the range of summation or integration.

[Trivial for $k=1$, otherwise a consequence of (H) with f replaced by $|fg^{1/k}|$, g by $|g|^{1/k'}$.]

(5) $$\left|\frac{1}{2\pi}\int fgh\,d\theta\right| \leqslant \left(\frac{1}{2\pi}\int|f^r h|\,d\theta\right)^{1/r}\left(\frac{1}{2\pi}\int|g^{r'}h|\,d\theta\right)^{1/r'} \quad (r>1).$$

(6) $$\left|\frac{1}{2\pi}\int fg\,d\theta\right| \leqslant \left(\frac{1}{2\pi}\int|f^h g^k|\,d\theta\right)^{1/r}\left(\frac{1}{2\pi}\int|f|^h d\theta\right)^{1/s}\left(\frac{1}{2\pi}\int|g|^k d\theta\right)^{1/t}$$

if $$h \geqslant 1,\quad k\geqslant 1,\quad h+k>hk,$$

where $$\frac{1}{r}=\frac{1}{h}+\frac{1}{k}-1,\quad \frac{1}{s}=1-\frac{1}{k},\quad \frac{1}{t}=1-\frac{1}{h}.$$

[For $|fg| = |f^h g^k|^{1/r}.(|f|^h)^{1/s}.(|g|^k)^{1/t}$].

(7) $$\left|\frac{1}{2\pi}\int f\,d\theta\right| \leqslant \left(\frac{1}{2\pi}\int|f|^k d\theta\right)^{1/k} \quad (k\geqslant 1).$$

This is the special case $g=1$ of (4). The parallel form is

(8) $$\left|\frac{1}{n}\Sigma a\right| \leqslant \left(\frac{1}{n}\Sigma|a|^k\right)^{1/k}$$

(8) is not homogeneous [nor is (7)]. If we suppress the factors $1/n$ the inequality becomes false; indeed, when the a's are non-negative, it becomes true with sign reversed, as is seen at once from Lemma α. (The result corresponding to this in the theory of integrals has little interest.)

2.6. We prove next (a's and b's not necessarily positive):

(1) $$(\Sigma|a|^k)^{1/k}-(\Sigma|b|^k)^{1/k} \leqslant (\Sigma|a+b|^k)^{1/k} \leqslant (\Sigma|a|^k)^{1/k}+(\Sigma|b|^k)^{1/k} \quad (k\geqslant 1).$$

(2) $$\Sigma|a|^\kappa-\Sigma|b|^\kappa \leqslant \Sigma|a+b|^\kappa \leqslant \Sigma|a|^\kappa+\Sigma|b|^\kappa \quad (0\leqslant\kappa\leqslant 1).$$

In each of (1) and (2) the left-hand inequality reduces to the right-hand one if we replace a by $a+b$ and b by $-b$. The right-hand inequality of (1) is (M). To prove that of (2) it is enough to show that $(a+b)^\kappa \leqslant a^\kappa+b^\kappa$ for $a, b\geqslant 0$: this reduces to

$$(1+x)^\kappa \leqslant 1+x^\kappa \quad (x\geqslant 0),$$

which is easily verified by differentiation (since $\kappa-1\leqslant 0$).

We can combine (1) and (2) as follows. For $\lambda>0$ let

$$\alpha=\alpha(\lambda)=\begin{cases}1 & (\lambda\leqslant 1)\\ 1/\lambda & (\lambda\geqslant 1)\end{cases}$$

Then

(3) $$(\Sigma|a|^\lambda)^\alpha-(\Sigma|b|^\lambda)^\alpha \leqslant (\Sigma|a+b|^\lambda)^\alpha \leqslant (\Sigma|a|^\lambda)^\alpha+(\Sigma|b|^\lambda)^\alpha,$$

or, what is the same thing,

(3′) $$\left|(\Sigma|a+b|^\lambda)^\alpha-(\Sigma|a|^\lambda)^\alpha\right| \leqslant (\Sigma|b|^\lambda)^\alpha.$$

The remaining results in this sub-section involve constant factors. Their value lies in application and the precise values of the constants are without importance. We have first a result roughly equivalent in application to (3′).

$$\Big|\Sigma|a+b|^{\lambda}-\Sigma|a|^{\lambda}\Big| \leqslant \Sigma|b|^{\lambda}+R_{\lambda}(a,\,b) \quad (\lambda>0), \tag{4}$$

where

$$R_{\lambda}(a,\,b)=\begin{cases}0 & (\lambda\leqslant 1)\\ A_{\lambda}\Big((\Sigma|a|^{\lambda})^{1-1/\lambda}(\Sigma|b|^{\lambda})^{1/\lambda}+(\Sigma|a|^{\lambda})^{1/\lambda}(\Sigma|b|^{\lambda})^{1-1/\lambda}\Big) & (\lambda>1).\end{cases}$$

This is proved [in (2)] if $\lambda\leqslant 1$; suppose then $\lambda>1$. The function

$$\{(1+x)^{\lambda}-1-x^{\lambda}\}/(x+x^{\lambda-1})$$

is bounded in $x\geqslant 0$ [consider $x\leqslant\frac{1}{2}$, $\frac{1}{2}\leqslant x\leqslant 2$, $x\geqslant 2$ separately] and non-negative, by Lemma α.

Hence, writing B for A_{λ}, we have for non-negative a, b,

$$0\leqslant(a+b)^{\lambda}-a^{\lambda}-b^{\lambda}\leqslant B(a^{\lambda-1}b+ab^{\lambda-1}),$$

$$0\leqslant\Sigma(a+b)^{\lambda}-\Sigma a^{\lambda}\leqslant\Sigma b^{\lambda}+B(\Sigma a^{\lambda-1}b+\Sigma ab^{\lambda-1}),$$

while finally

$$\Sigma a^{\lambda-1}b\leqslant(\Sigma a^{\lambda})^{1-1/\lambda}(\Sigma b^{\lambda})^{1/\lambda},\quad \Sigma ab^{\lambda-1}\leqslant(\Sigma a^{\lambda})^{1/\lambda}(\Sigma b^{\lambda})^{1-1/\lambda}.$$

The case $a\geqslant 0$, $b\geqslant 0$, is therefore disposed of. Consider now the general case: we have

$$\Delta=\Sigma|a+b|^{\lambda}-\Sigma|a|^{\lambda}\leqslant\Sigma|b|^{\lambda}+R_{\lambda}(a,\,b) \tag{5}$$

a fortiori from the positive case. On the other hand, by the same argument,

$$\begin{aligned}-\Delta&=\Sigma|a^{\lambda}|-\Sigma|a+b|^{\lambda}\\ &=\Sigma|(a+b)+(-b)|^{\lambda}-\Sigma|a+b|^{\lambda}\leqslant\Sigma|-b|^{\lambda}+R_{\lambda}(a+b,\,-b).\end{aligned} \tag{6}$$

If Δ is positive (5) gives us what we want. If, on the other hand, Δ is negative, then $R_{\lambda}(a+b,-b)\leqslant R_{\lambda}(a,\,b)$, [since $1-1/\lambda>0$] and (6) gives what we want. This completes the proof.

Next we have two simpler results. For $\lambda>0$

$$\Sigma|a+b|^{\lambda}\leqslant A_{\lambda}(\Sigma|a|^{\lambda}+\Sigma|b|^{\lambda}), \tag{7}$$

$$(\Sigma|a+b|^{\lambda})^{1/\lambda}\leqslant A_{\lambda}\{(\Sigma|a|^{\lambda})^{1/\lambda}+(\Sigma|b|^{\lambda})^{1/\lambda}\}, \tag{8}$$

with extensions to more than two sets.

For $\quad |a+b|^{\lambda}\leqslant\{2\operatorname{Max}(|a|,|b|)\}^{\lambda}\leqslant 2^{\lambda}(|a|^{\lambda}+|b|^{\lambda}).$

Thus (7) is true with $A_{\lambda}=2^{\lambda}$. Further

$$\Sigma|a+b|^{\lambda}\leqslant 2^{\lambda}.2\operatorname{Max}(\Sigma|a|^{\lambda},\ \Sigma|b|^{\lambda}),$$

and so $\quad (\Sigma|a+b|^{\lambda})^{1/\lambda} \leqslant 2^{1+1/\lambda}\,\mathrm{Max}\{(\Sigma|a|^{\lambda})^{1/\lambda},\ (\Sigma|b|^{\lambda})^{1/\lambda}\}$

$$\leqslant 2^{1+1/\lambda}\{(\Sigma|a|^{\lambda})^{1/\lambda}+(\Sigma|b|^{\lambda})^{1/\lambda}\},$$

which proves (8).

We now prove that, for $\lambda>0$,

(9) *if* $\int_E |f-f_n|^{\lambda}\,d\theta \to 0$ *as* $n\to\infty$, *then* $\int_E |f_n|^{\lambda}\,d\theta \to \int_E |f|^{\lambda}\,d\theta$;

(10) *if* $\int_E |f-f_n|^{\lambda}\,d\theta \to 0$, $\int_E |f^*-f_n|^{\lambda}\,d\theta\to 0$, *then* $f^*\equiv f$ *in* E.

In fact, by (3)′ (for integrals),

$$\left(\int|f_n|^{\lambda}\,d\theta\right)^{a}-\left(\int|f|^{\lambda}\,d\theta\right)^{a}\Big|\leqslant\left(\int|f_n-f|^{\lambda}\,d\theta\right)^{a}\to 0$$

provided $\int|f|^{\lambda}\,d\theta$ is finite. If the last integral is not finite we can conclude that for any fixed N

$$\int|f_n|^{\lambda}d\theta\geqslant\int|[f_n]_N|^{\lambda}\,d\theta\to\int|[f]_N|^{\lambda}\,d\theta$$

and so $\int|f_n|^{\lambda}\,d\theta\to\infty$, since the last expression tends to ∞ with N. Thus (9) is true whether $\int|f|^{\lambda}d\theta$ is finite or infinite.

It is not difficult to deduce (9) also from (4).

For (10) we have $f^*-f=(f_n-f)+(f^*-f_n)$, and so

$$\int|f^*-f|^{\lambda}\,d\theta\leqslant A_{\lambda}\left\{\int|f_n-f|^{\lambda}d\theta+\int|f_n-f^*|^{\lambda}\,d\theta\right\}$$

by (7). Since the right-hand side tends to zero the left side is equal to zero, and the non-negative integrand is equivalent to zero.

2.7. *The means* $M_{\mu}(f)$.

2.71. We define, for any finite $\mu\neq 0$,

$$M_{\mu}(f)=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^{\mu}\,d\theta\right)^{1/\mu},$$

$$\Lambda_{\mu}(f)=\log M_{\mu}(f).$$

For $\mu=+\infty$ we define $M_{\infty}=e^{\Lambda_{\infty}}$ as the greatest number M such that for every ϵ $|f|>M-\epsilon$ in a set of positive measure, or as ∞ if no M exists. We call this number also $\mathrm{Max}(|f|)$ or the maximum of $|f|$: equivalent functions have the same maximum, and for any f there exists on equivalent $f^*(=[f]_M)$, of which M is the maximum in the ordinary sense. Similarly we define $\mathrm{Min}|f|=M_{-\infty}=e^{\Lambda_{-\infty}}$ as the least m such

that $|f|<m+\epsilon$ in a set of positive measure. For a continuous f M and m are, of course, $\operatorname{Max}|f|$ and $\operatorname{Min}|f|$ in the usual sense. Finally we define

$$\Lambda_0=\log M_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f|\,d\theta.$$

This integral has a definite value (possibly $+\infty$ or $-\infty$) unless the integral over the positive values of the integrand and that over the negative values are both infinite. In the latter case we regard Λ_0 as taking *all* values from $-\infty$ to $+\infty$, and interpret statements about it in the obvious way (*e.g.* $\Lambda_\mu \geqslant \Lambda_0$ would mean $\Lambda_\mu=+\infty$). $\mu=0$ is a genuinely exceptional suffix, but the gloss enables us to remove those points of difference that are merely trivial.

We observe that

(1) $$\Lambda_{-\mu}(f)=-\Lambda_\mu(1/f)\quad(-\infty\leqslant\mu\leqslant+\infty),$$

a result which enables us to infer propositions about negative μ from those for positive μ.

2.72. We prove next :

(2) $$\Lambda_\mu\to\Lambda_{+\infty}\ \textit{as}\ \mu\to+\infty,\quad \Lambda_\mu\to\Lambda_{-\infty}\ \textit{as}\ \mu\to-\infty.$$

It is enough, by (1), to prove the first. If $M_\infty<+\infty$ there exists an f^* such that

$$|f|\equiv|f^*|\leqslant M_\infty,$$

and so
$$M_\mu(f)=M_\mu(f^*)\leqslant M_\infty.$$

On the other hand, $|f|>M_\infty-\epsilon$ in a set E of measure $\delta>0$,

$$M_\mu(f)\geqslant\left\{\frac{\delta}{2\pi}(M_\infty-\epsilon)^\mu\right\}^{1/\mu},$$

$$\varliminf_{\mu\to\infty}M_\mu\geqslant M_\infty-\epsilon,\quad \varliminf M_\mu\geqslant M_\infty.$$

Hence $M_\mu\to M_\infty$. If $M_\infty=\infty$ we have, for an arbitrarily large K, $|f|>K$ in a set of positive measure δ,

$$M_\mu\geqslant\left(\frac{\delta}{2\pi}K^\mu\right)^{1/\mu},\quad \varliminf M_\mu\geqslant K,\quad \lim M_\mu=\infty.$$

2.73. We show next : Λ_μ *is an increasing function of* μ (*in the wide sense*). We have to show that $\Lambda_{\mu_2}\geqslant\Lambda_{\mu_1}$ if $\mu_2>\mu_1$. If $\mu_1>0$ this follows from § 2.5 (7) [with $|f|^{\mu_1}$ for f, $k=\mu_2/\mu_1$], and (1) above then shows that it is true also for $\mu_2<0$. (Incidentally we see that

$$M_{+0}=\lim_{\mu\to+0}M_\mu$$

exists, and similarly for M_{-0}.) It is therefore sufficient to prove

(1) $$\Lambda_\mu \geqslant \Lambda_0 \quad (\mu > 0),$$

and to this end we may suppose $\Lambda_\mu < +\infty$, $M_\mu < \infty$. Then, since

$$\overset{+}{\log}|f| = \operatorname{Max}(\log|f|, 0) \leqslant A_\mu |f|^\mu,$$

we have $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \overset{+}{\log}|f|\, d\theta < \infty,$$

and *a fortiori* $$\Lambda_0 = \frac{1}{2\pi}\int \log|f|\, d\theta < \infty.$$

[This means incidentally that Λ_0 has a definite value less than ∞.] Supposing, as we may, that $\Lambda_0 > -\infty$, we have now

$$|f|^\mu = \exp(\mu \log|f|) \geqslant 1+\mu\log|f|,$$

$$M_\mu^\mu \geqslant 1+\mu\Lambda_0,$$

and so $$M_{+0} = \lim_{\mu\to+0} M_\mu \geqslant \lim(1+\mu\Lambda_0)^{1/\mu} = \exp\Lambda_0 = M_0.$$

Since M_μ is increasing for $\mu > 0$ this gives $\Lambda_\mu \geqslant \Lambda_0$ $(\mu > 0)$.

2.74. *Continuity of* Λ_μ, M_μ. We define now (as above)

$$\Lambda_{\pm 0} = \lim_{\mu\to\pm 0}\Lambda_\mu, \quad M_{\pm 0} = \lim_{\mu\to\pm 0} M_\mu.$$

As above we have

(1) $$\Lambda_{-0} \leqslant \Lambda_0 \leqslant \Lambda_{+0}, \quad M_{-0} \leqslant M_0 \leqslant M_{+0}.$$

We show now: *If $\mu_1 > 0$ and Λ_{μ_1} is finite, then Λ_μ is continuous for every μ in $0 < \mu < \mu_1$, and is continuous on the left at $\mu = \mu_1$.*

In the first place Λ_μ is finite in $0 < \mu < \mu_1$ since $\Lambda_\mu \leqslant \Lambda_{\mu_1} < +\infty$ and $\Lambda_\mu = -\infty$ gives $M_\mu = 0$, $f \equiv 0$, $M_{\mu_1} = 0$, $\Lambda_{\mu_1} = -\infty$. Let now μ tend to a μ_0, where $0 < \mu_0 \leqslant \mu_1$, but only from below if $\mu_0 = \mu_1$. Then we have

$$|f|^\mu \leqslant 1+|f|^{\mu_1},$$

a function of class L independent of μ. By a well known theorem [proved below as Theorem 10],

$$\lim_{\mu\to\mu_0}\int |f|^\mu d\theta = \int \lim|f|^\mu d\theta = \int |f|^{\mu_0} d\theta,$$

or $$M_\mu \to M_{\mu_0}.$$

Suppose now further that Λ_0 is finite; we shall show that Λ_μ *is continuous in* $+0 \leqslant \mu \leqslant \mu_1 - 0$. It is enough to prove that $\Lambda_0 = \Lambda_{+0}$ or, on

account of (1), that $\Lambda_{+0} \leqslant \Lambda_0$. Suppose first that $|f|$ is bounded below by a positive constant, which we may suppose, on grounds of homogeneity, to be 1. Then $\log |f| \geqslant 0$ and

$$|f|^\mu \leqslant 1+\mu \log|f| + (\mu \log |f|)^2(1+|f|^{\frac{1}{2}\mu_1}) \quad (\mu \leqslant \tfrac{1}{2}\mu_1),$$

(2) $$M_\mu^\mu \leqslant 1+\mu\Lambda_0+\mu^2 J,$$

where $$J = \frac{1}{2\pi}\int (\log|f|)^2(1+|f|^{\frac{1}{2}\mu_1})\, d\theta < \infty$$

since $M_{\mu_1} < \infty$. Hence

$$\lim_{\mu \to +0} M_\mu \leqslant \lim (1+\mu\Lambda_0+\mu^2 J)^{1/\mu} = \exp \Lambda_0,$$

the desired result. Finally, no longer supposing $|f|$ bounded below, let $f_n = \operatorname{Max}(|f|, n^{-1})$. Then

$$\begin{aligned}\Lambda_{+0}(f) &\leqslant \Lambda_{+0}(f_n)\\ &\leqslant \Lambda_0(f_n)\end{aligned}$$

by the bounded case. This being true for all n we have†

$$\Lambda_0(f) = \lim_{n\to\infty} \Lambda_0(f_n) \geqslant \Lambda_{+0}(f).$$

This completes the proof.

We conclude with one or two simple observations.

(3) *If* $0 \leqslant \mu < \mu_1$, $\Lambda_{\mu_1} < \infty$, *and* $\Lambda_\mu = -\infty$, *then* $\Lambda_{\mu+0} = -\infty$.

For $$\Lambda_{\mu+0}(f) \leqslant \Lambda_{\mu+0}(f_n) = \Lambda_\mu(f_n),$$

and this tends to $-\infty$ whether $\mu > 0$ or $\mu = 0$.

(4) *The relations* $\Lambda_\mu < \infty$, $\Lambda_{\mu+0} = +\infty$ *are (simultaneously) possible for a* $\mu > 0$.

(5) $-\infty = \Lambda_{-0} < \Lambda_0 < \Lambda_{+0} = +\infty$ *is possible.*

The function $f = \{|\theta|(\log|\theta|)^2\}^{-1/\mu}$ has the property (4), and $f = \exp(|\theta|^{-\frac{1}{2}} \operatorname{sgn} \theta)$ has the property (5).

The results of the subsection, collected and extended to negative μ by means of (1), are given in (10) to (18) of Theorem 1 below. The facts are simpler than the arguments, and the best classification involves some rearrangement, which the reader may profitably verify.

† By Theorem 11 Cor. below, f_n being monotonic.

2 . 75. *Convexity.*

LEMMA β. *Suppose* $0 < \alpha \leqslant \beta \leqslant \gamma$, $\alpha < \gamma$, *and* $M_\gamma(f) < \infty$. *Then*

(1) $$M_\beta \leqslant M_\alpha^\vartheta M_\gamma^{1-\vartheta}, \quad \textit{where} \ \vartheta = \frac{\alpha(\gamma-\beta)}{\beta(\gamma-\alpha)}, \quad 1-\vartheta = \frac{\gamma(\beta-\alpha)}{\beta(\gamma-\alpha)}.$$

In other language, $\log(M_\lambda^\lambda)$ *is a convex function of* λ.†

The result is trivial unless $\alpha < \beta < \gamma$, which we suppose. Then $0 < \vartheta < 1$, and if $r = \alpha/(\beta\vartheta) = (\gamma-\alpha)/(\gamma-\beta) > 1$, we have

$$|f|^\beta = |f|^{\alpha/r}|f|^{\gamma/r'},$$

(2) $$M_\beta^\beta = \frac{1}{2\pi}\int |f|^\beta d \leqslant \left(\frac{1}{2\pi}\int |f|^\alpha d\right)^{1/r}\left(\frac{1}{2\pi}\int |f|^\gamma d\right)^{1/r'} = M_\alpha^{\alpha/r} M_\gamma^{\gamma/r'},$$

$$M_\beta \leqslant M_\alpha^{\alpha/(\beta r)} M_\gamma^{\gamma/(\beta r')} = M_\alpha^\vartheta M_\gamma^{1-\vartheta}.$$

We show next that equality can occur in (1), with $\alpha < \beta < \gamma$, only if there exists a set $E \subset E_0$ such that $|f| \equiv c$ in E and $f \equiv 0$ in CE, c being a positive constant, and that in this case (1) is true, with the sign of equality, for all α, β, γ [satisfying $0 < \alpha \leqslant \beta \leqslant \gamma$].

The last part is evident. If now equality holds in (1) it holds in (2). Hence $|f|^\alpha \equiv c|f|^\gamma$, by (2) of § 2 . 1, and this leads at once to the first part.

As a corollary of Lemma β we have: If $\lambda_1 < \lambda_2$, $M_{\lambda_1}(f_n)$ tends to zero as $n \to \infty$, and $M_{\lambda_2}(f_n)$ is bounded, then $M_\lambda(f_n) \to 0$ for $\lambda < \lambda_2$.

Results parallel to these hold also, of course, for means of sums. Thus, if $S_\alpha = (\Sigma |a|^\alpha)^{1/\alpha}$ and we suppose in the first instance that there are n a's, we have

$$n^{-1/\beta} S_\beta \leqslant (n^{-1/\alpha} S_\alpha)^\vartheta (n^{-1/\gamma} S_\gamma)^{1-\vartheta}.$$

We may, however, drop the n-factors on account of homogeneity, and may then suppose the a's infinite in number by a passage to the limit. Thus

$$S_\beta \leqslant S_\alpha^\vartheta S_\gamma^{1-\vartheta},$$

where ϑ is the number in (1), and equality can hold, for $\alpha < \beta < \gamma$ and $S_\alpha < \infty$ [in which case $S_\gamma \leqslant S_\alpha < \infty$ by Lemma α] only if all a's are zero except a finite number, and for these $|a| =$ constant. If this happens, then equality does hold for α, β, γ.

† $\phi(x)$ is a convex function of x if, for any interval $\alpha \leqslant x \leqslant \beta$ $\phi(x) \leqslant \psi(x)$, where $\psi(x)$ is the linear function of x which agrees with $\phi(x)$ at $x = \alpha$ and $x = \beta$. Or, if "the arc lies below the chord".

2 . 76. The three-term inequalities in § 2 . 75 are homogeneous. We consider now two-term inequalities; these are non-homogeneous, and "means" and "sums (integrals)" behave differently. We have seen already that $M_\alpha(f) \leqslant M_\gamma(f)$ $(\alpha \leqslant \gamma)$, and Lemma α shows that $S_\alpha \geqslant S_\gamma$ $(\alpha \leqslant \gamma)$. [Thus means and sums are monotonic in opposite senses. Neither monotony is implied, of course, by the convexity.] We proceed to consider the conditions for equality in the (non-trivial) case $\alpha < \gamma$. The results are :

(1) *If $\alpha < \gamma$ and $M_\gamma < \infty$, then $M_\alpha = M_\gamma$ if and only if $|f| \equiv c$, where c is a constant, and then equality holds for all α, γ.*

(2) *If $\alpha < \gamma$ and $S_\alpha < \infty$, then $S_\alpha = S_\gamma$ if and only if all a's are zero but one, and then equality holds for all α, γ.*

Let $\beta = \frac{1}{2}(\alpha+\gamma)$. Then, by Lemma β, $M_\beta \leqslant M_\alpha$, $S_\beta \leqslant S_\gamma$, and so $M_\alpha = M_\beta = M_\gamma$, $S_\alpha = S_\beta = S_\gamma$ (if the extremes are equal). These require respectively $|f| \equiv \phi$ when ϕ is everywhere 0 or c; $|a| = c$ for a finite number ν of a's, the rest being zero. If E is the set in which $\phi = c$ we have

$$c(mE/2\pi)^{1/\alpha} = M_\alpha = M_\gamma = c(mE/2\pi)^{1/\gamma},$$

whence $mE = 2\pi$ if $c \neq 0$. Similarly

$$\nu^{1/\alpha} c = S_\alpha = S_\gamma = \nu^{1/\gamma} c,$$

and $\nu = 0$ or 1 if $c \neq 0$. These facts establish (1) and (2).

2 . 77. We conclude by determining the limits to which the monotonic function S_α tends as $\alpha \to 0$ and $\alpha \to \infty$.

(1) *If $S_\alpha < \infty$ for all $\alpha > 0$ then $S_\alpha \to \infty$ as $\alpha \to 0$ unless there is only one a different from zero, in which case S_α is the same for all α.*

For if $a_1, a_2 \neq 0$ $|a_1|^\alpha + |a_2|^\alpha > \frac{3}{2}$ for $\alpha < \alpha_0$ and $S_\alpha > (\frac{3}{2})^{1/\alpha} \to \infty$.

(2) *If $S_\alpha < \infty$ for some sufficiently large α, then $S_\alpha \to \operatorname{Max}|a|$ as $\alpha \to \infty$.*

Clearly $|a| = \mu = \operatorname{Max}|a| > 0$ can occur for at most a finite number, ν say, of a's, otherwise $\Sigma|a|^\alpha$ would diverge for every α. Hence

$$\left(\frac{S_\alpha}{\mu}\right)^\alpha = \nu + \Sigma b_n^\alpha,$$

where every $b_n < 1$. Further, the last series is convergent for $\alpha \geqslant \alpha_0$, say, otherwise $S_\alpha = \infty$ for every α. But then Σb_n^α is uniformly convergent for $\alpha \geqslant \alpha_0$, and so tends to $\Sigma \lim b_n^\alpha = \Sigma 0 = 0$ as $\alpha \to \infty$. Thus

$$(S_\alpha/\mu)^\alpha \to \nu, \quad S_\alpha \to \mu.$$

2 . 81. We now sum up the results obtained so far in a number of separate theorems. We do not always give the parallel results for both sums and integrals, and shall treat as sufficient a reference to a "sum" result where the application requires the "integral" one.

In Theorems 1 to 4 we use the index conventions :

$$\mu \text{ real},\ \lambda > 0,\ k \geqslant 1,\ r > 1;\ \ 0 < \kappa \leqslant 1,\ 0 < \rho < 1;\ \ t' = t/(t-1).$$

Other letters denote in general arbitrary complex numbers. We recall the definitions

$$M_\mu(f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^\mu d\theta\right)^{1/\mu} \ (\mu \neq 0), \quad \Lambda_\mu(f) = \log M_\mu(f),$$

$$\Lambda_0(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f|\, d\theta,$$

subject to the gloss explained in § 2 . 71.

$$M_\infty = \operatorname{Max}|f|, \quad M_{-\infty} = \operatorname{Min.}|f|.\dagger$$

$$S_\lambda^\lambda = \Sigma|a|^\lambda.$$

THEOREM 1. *We have the following inequalities, integrals being taken from* $-\pi$ *to* π, *unless otherwise stated.*

(1) $$\left|\frac{1}{2\pi}\int fg\, d\theta\right| \leqslant \left(\frac{1}{2\pi}\int |f|^r d\theta\right)^{1/r} \left(\frac{1}{2\pi}\int |g|^{r'} d\theta\right)^{1/r'} = M_r(f)\, M_{r'}(g).$$

More generally

(2) $$\left|\frac{1}{2\pi}\int fgh \ldots d\theta\right| \leqslant M_{r_1}(f)\, M_{r_2}(g) \ldots \quad \textit{if} \ \ \Sigma\frac{1}{r} = 1.$$

(3) (2) *remains true subject to* $\Sigma \frac{1}{r} \leqslant 1$.

(4) $$\left|\frac{1}{2\pi}\int fg\, d\theta\right|^k \leqslant \left(\frac{1}{2\pi}\int |f^k g|\, d\theta\right) \left(\frac{1}{2\pi}\int |g|\, d\theta\right)^{k-1}.$$

(5) $$\left|\frac{1}{2\pi}\int fgh\, d\theta\right| \leqslant \left(\frac{1}{2\pi}\int |f^r h|\, d\theta\right)^{1/r} \left(\frac{1}{2\pi}\int |g^{r'} h|\, d\theta\right)^{1/r'}.$$

(6) $$\left|\frac{1}{2\pi}\int fg\, d\theta\right| \leqslant \left(\frac{1}{2\pi}\int |f^h g^k|\, d\theta\right)^{1/r} \left(\frac{1}{2\pi}\int |f|^h d\theta\right)^{1/s} \left(\frac{1}{2\pi}\int |g|^k d\theta\right)^{1/t}$$

† In the modified sense of § 2 . 71.

for any h, k *satisfying* $h \geqslant 1$, $k \geqslant 1$, $h+k > hk$, *where*

$$\frac{1}{r} = \frac{1}{h} + \frac{1}{k} - 1, \quad \frac{1}{s} = 1 - \frac{1}{k}, \quad \frac{1}{t} = 1 - \frac{1}{h}.$$

(7) $$|ab| \leqslant \frac{|a|^r}{r} + \frac{|b|^{r'}}{r'}.$$

(8) *In* (1), (2), (4), (5), (6) *we may suppress the factors* $\frac{1}{2\pi}$ *and suppose the integrals taken over any set* E, *of finite or infinite measure.*
(9) *When everything is positive the inequality* (1) *is true with reversed sign when* r *is replaced by a non-zero* $\mu < 1$.

We have further the following results, supposing [*in* (10) *to* (18)] *that* $|f|$ *is not almost everywhere zero or almost everywhere* ∞.

(10) $\Lambda_\mu(f)$, $M_\mu(f)$ *are increasing functions of* μ (*in the wide sense*). *In particular* $M_{\mu+0}(f)$, $M_{\mu-0}(f)$ *exist.*

(11) $$M_\mu(f) \to M_\infty(f) = \mathrm{Max}\,|f| \text{ as } \mu \to \infty,$$
$$M_\mu(f) \to M_{-\infty}(f) = \mathrm{Min}\,|f| \text{ as } \mu \to -\infty.$$

(12) *If* Λ_{μ_1}, Λ_{μ_2} *are finite and* $\mu_1 < \mu_2$, *then* Λ_μ *is continuous in* $\mu_1 \leqslant \mu \leqslant \mu_2$. [Either μ_1 or μ_2 may, of course, be 0.] *Suppose* $\mu_1 \neq 0$ *and* Λ_{μ_1} *is finite, then* Λ_μ *is continuous in the interval* $(0, \mu_1)$ *taken closed at* μ_1 *and open at* 0. *Further* $\Lambda_\mu \to \Lambda_0$ *as* $\mu \to 0$ *in* $(0, \mu_1)$ [but Λ_0 may be $\pm\infty$]; *in particular, if* $\mu_1 > 0$, *then* M_μ *is continuous in* $0 \leqslant \mu \leqslant \mu_1$.

(13) *At a fixed point* $\mu > 0$ *the alternatives are* :

(i) $-\infty < \Lambda_\mu < \infty$, $\Lambda_{\mu-0} = \Lambda_\mu$, $\Lambda_{\mu+0}$ *is either* Λ_μ *or* ∞.

(ii) $\Lambda_\mu = \Lambda_{\mu+0} = \infty$, $-\infty < \Lambda_{\mu-0} \leqslant \infty$.

(14) *At a fixed* $\mu < 0$ *the alternatives are* :

(i) $-\infty < \Lambda_\mu < \infty$, $\Lambda_{\mu+0} = \Lambda_\mu$, $\Lambda_{\mu-0}$ *is either* Λ_μ *or* $-\infty$.

(ii) $\Lambda_\mu = \Lambda_{\mu-0} = -\infty$, $-\infty \leqslant \Lambda_\mu < \infty$.

(15) *At* $\mu = 0$ *the alternatives are* :

(i) $-\infty < \Lambda_0 < \infty$, $\Lambda_{+0} = \Lambda_0$ *or* $+\infty$, $\Lambda_{-0} = \Lambda_0$ *or* $-\infty$.

(ii) $\Lambda_{-0} = \Lambda_0 = \Lambda_{+0} = \infty$.

(iii) $\Lambda_{-} = \Lambda_0 = \Lambda_{+0} = -\infty$.

(16) *In* (13) *to* (15) *all possibilities left open can actually occur.*

(17) Λ_μ *cannot have a finite discontinuity at any* μ.

(18) *Except at* $\mu=0$ *one or other of* $\Lambda_{\mu+0}$, $\Lambda_{\mu-0}$ *is equal to* Λ_μ.

(19) *Let* $0<\alpha<\beta<\gamma$, $M_\gamma(f)<\infty$. *Then*

$$M_\beta \leqslant M_\alpha^{\vartheta} M_\gamma^{1-\vartheta}, \quad \textit{where} \quad \vartheta=\frac{\alpha(\gamma-\beta)}{\beta(\gamma-\alpha)}, \quad 1-\vartheta=\frac{\gamma(\beta-\alpha)}{\beta(\gamma-\alpha)}.$$

(20) *Equality occurs in* (19) *if and only if* $|f|\equiv c$ *in some set* E *and* $f\equiv 0$ *in* CE, *and then it occurs for all* α, β, γ *(satisfying* $0<\alpha<\beta<\gamma$).

(21) *If* $0<\alpha<\gamma$ *and* $M_\gamma(f)<\infty$, *then* $M_\alpha=M_\gamma$ *if and only if* $|f|\equiv c$, *and then all* M_α *are equal.*

(22) *If* $\lambda_1<\lambda_2$, $M_{\lambda_1}(f_n)\to 0$ *as* $n\to\infty$, *and* $M_{\lambda_2}(f_n)$ *is bounded as* $n\to\infty$, *then* $M_\lambda(f_n)\to 0$ *for all* $\lambda<\lambda_2$.

(23) S_λ *is a decreasing function of* λ *(in the wide sense).*

(24) *If* $S_\lambda<\infty$ *for all* $\lambda>0$, *then* $S_\lambda\to\infty$ *as* $\lambda\to 0$, *only one* a *different from zero, in which case* S_λ *is the same for all* λ.†

(25) *If* $S_\lambda<\infty$ *for some sufficiently large* λ, *then* $S_\lambda\to\text{Max}\,|a|$ *as* $\lambda\to\infty$.

(26) *Let* $0<\alpha<\beta<\gamma$, $S_\alpha<\infty$. *Then*

$$S_\beta \leqslant S_\alpha^{\vartheta} S_\gamma^{1-\vartheta}, \quad \textit{where} \quad \vartheta=\frac{\alpha(\gamma-\beta)}{\beta(\gamma-\alpha)}, \quad 1-\vartheta=\frac{\gamma(\beta-\alpha)}{\beta(\gamma-\alpha)}.$$

(27) *Equality occurs in* (26) *only if all* a*'s are zero except a finite number for which* $|a|$ *is constant, and then equality occurs for all* α, β, γ.

(28) *If* $0<\alpha<\gamma$ *and* $S_\alpha<\infty$, *then* $S_\alpha=S_\gamma$ *only if all* a*'s but one are zero, and then all* S_α *are equal.*‡

THEOREM 2.

(1) $$(\Sigma|a+b|^k)^{1/k} \leqslant (\Sigma|a|^k)^{1/k}+(\Sigma|b|^k)^{1/k} \quad (k\geqslant 1)$$

---

† If n is the number of a's, then $n^{-1/\lambda}S_\lambda\to(\Pi a)^{1/n}$: *this* result is the analogue of "$M_\lambda\to M$ as $\lambda\to 0$ (provided some $M_\lambda<\infty$)".

‡ The analogue for integrals (as opposed to means) is: if

$$\left(\int_E |f|^\alpha\, d\theta\right)^{1/\alpha} = \left(\int_E |f|^\gamma\, d\theta\right)^{1/\gamma}$$

then $f\equiv 0$ except in a sub-set of E of measure unity, in which $|f|\equiv c$.

More generally

(2) $$\left(\sum_n \left|\sum_m a_{m,n}\right|^k\right)^{1/k} \leqslant \sum_m \left(\sum_n |a_{m,n}|^k\right)^{1/k}.$$

(3) $$\left(\int_E \left|\sum_m f_m(x)\right|^k dx\right)^{1/k} \leqslant \sum_m \left(\int_E |f_m(x)|^k dx\right)^{1/k}.$$

(4) $$\left(\sum_n \left|\int_E f_n(x)\,dx\right|^k\right)^{1/k} \leqslant \int_E \left(\sum_n |f_n(x)|^k\right)^{1/k} dx.$$

(5) $$\left(\int \left|\int f(x, y)\,dy\right|^k dx\right)^{1/k} \leqslant \int \left(\int |f(x, y)|^k dx\right)^{1/k} dy.$$

(2) *may be generalized to*

(6) $$\left(\sum_n \left|\sum_m c_m a_{m,n}\right|^k\right)^{1/k} \leqslant \sum_m |c_m| \left(\sum_n |a_{m,n}|^k\right)^{1/k},$$

(5) *may be generalized to*

(7) $$\left(\int \left|\int f(x, y)\,\phi(y)\,dy\right|^k |\psi(x)|\,dx\right)^{1/k}$$
$$\leqslant \int \left(\int |f(x, y)|^k |\psi(x)|\,dx\right)^{1/k} |\phi(y)|\,dy,$$

and similar extensions may be given to (3) *and* (4).

(8) *Equality occurs in* (1) *if and only if all* b*'s are zero or else* $b = ca$, *where* c *is a positive constant. Equality occurs in* (5) *if and only if* $\operatorname{sgn} f(x, y)$ *is constant where it is not zero, and* $f(x, y) = F(x)\,G(y)$, *p.p. in* x *and p.p. in* y.

(9) *When everything (but the index) is positive the inequalities* (1) *to* (5) *are true with the reverse sign if* k *is replaced by any non-zero* $\mu < 1$.

(10) $$(\Sigma|a|^k)^{1/k} - (\Sigma|b|^k)^{1/k} \leqslant (\Sigma|a+b|^k)^{1/k} \leqslant (\Sigma|a|^k)^{1/k} + (\Sigma|b|^k)^{1/k} \quad (k \geqslant 1)$$

(11) $$\Sigma|a|^\kappa - \Sigma|b|^\kappa \leqslant \Sigma|a+b|^\kappa \leqslant \Sigma|a|^\kappa + \Sigma|b|^\kappa \quad (0 \leqslant \kappa \leqslant 1).†$$

(12) $$\left|(\Sigma|a+b|^\lambda)^\alpha - (\Sigma|a|^\lambda)^\alpha\right| \leqslant (\Sigma|b|^\lambda)^\alpha, \quad \alpha = \begin{cases} 1 & (\lambda \leqslant 1) \\ 1/\lambda & (\lambda \geqslant 1) \end{cases}.$$

(13) $$(\Sigma|a+b+c+\ldots|^\lambda)^\alpha \leqslant (\Sigma|a|^\lambda)^\alpha + (\Sigma|b|^\lambda)^\alpha + (\Sigma|c|^\lambda)^\alpha + \ldots.$$

(14) $$\left|(\Sigma|a+b+c+\ldots|^\lambda)^\alpha - (\Sigma|a|^\lambda)^\alpha\right| \leqslant (\Sigma|b|^\lambda)^\alpha + (\Sigma|c|^\lambda)^\alpha + \ldots.$$

(15) $$\left|\Sigma|a+b|^\lambda - \Sigma|a|^\lambda\right| \leqslant \Sigma|b|^\lambda + R,$$

† For non-negative a, b there exist the following companions to (10) and (11):
$$\Sigma(a+b)^k \geqslant \Sigma a^k + \Sigma b^k, \quad \left\{\Sigma(a+b)^k\right\}^{1/k} \geqslant (\Sigma a^k)^{1/k} + (\Sigma b^k)^{1/k}.$$
The first is an immediate corollary of Lemma α, the second is a case of (9).

where $R = 0$ *if* $\lambda \leqslant 1$, *and*

$$R = A_\lambda\left((\Sigma|a|^\lambda)^{1-1/\lambda}(\Sigma|b|^\lambda)^{1/\lambda}+(\Sigma|a|^\lambda)^{1/\lambda}(\Sigma|b|^\lambda)^{1-1/\lambda}\right)$$

if $\lambda > 1$.

(16) $$\Sigma|a+b|^\lambda \leqslant A_\lambda(\Sigma|a|^\lambda+\Sigma|b|^\lambda).$$

(17) $$(\Sigma|a+b|^\lambda)^{1/\lambda} \leqslant A_\lambda\{(\Sigma|a|^\lambda)^{1/\lambda}+(\Sigma|b|^\lambda)^{1/\lambda}\}.$$

(18) *If* $\int_E |f-f_n|^\lambda d\theta \to 0$ *as* $n\to\infty$, *then* $\int_E |f_n|^\lambda d\theta \to \int_E |f|^\lambda d\theta$, *where the right-hand side may be finite or infinite.*

(19) *If* $\int_E |f-f_n|^\lambda d\theta \to 0$ *and* $\int_E |f^*-f_n|^\lambda d\theta \to 0$, *then* $f^* \equiv f$.

2.82. To the preceding theorems we add two more, of which the first constitutes a kind of converse of (H). They arise naturally out of our considerations of the case of equality in (H).

THEOREM 3. *Suppose that* $r > 1$ *and* E *is any set of finite non-zero measure. Suppose that* (f *being any complex function*)

(1) $$\left|\frac{1}{mE}\int_E fg\,d\theta\right| \leqslant UV$$

for every bounded function g *such that*

(2) $$\frac{1}{mE}\int_E |g|^{r'} d\theta = V^{r'} > 0.$$

Then

(3) $$\frac{1}{mE}\int_E |f|^r d\theta \leqslant U^r.$$

If f *is real it is sufficient for* (3) *that* (1) *should hold for every real bounded* g *satisfying* (2).

We may suppose $U \geqslant 0$. If now we take g to be a suitable constant in $[E]_N$ and zero elsewhere [the constant being chosen so that (2) is satisfied], our hypothesis asserts, among other things, that f is integrable, and so measurable, in $[E]_N$, and so measurable in E.

It is sufficient to show that

(4) $$\frac{1}{mE}\int_E |[f]_N|^r d\theta \leqslant U^r.$$

If the left side is zero there is nothing to prove. If not, choose

$$g = t\,|[f]_N|^{r-1}\,\overline{\operatorname{sgn} f}\dagger$$

and the constant t so that (2) is satisfied, *i.e.* so that

$$t^{r'}\frac{1}{mE}\int_E |[f]_N|^r\,d\theta = V^r. \tag{5}$$

Since g is bounded (1) holds, by hypothesis. This gives

$$t\frac{1}{mE}\int_E |[f]_N|^r\,d\theta = \frac{1}{mE}\int_E [f]_N g\,d\theta \leqslant \frac{1}{mE}\left|\int_E fg\,d\theta\right| \leqslant UV. \tag{6}$$

Eliminating t between (5) and (6) we obtain (4).

THEOREM 4. *Suppose that, all letters denoting non-negative numbers, $\Sigma a_n b_n$ is convergent for every set of b's for which $\Sigma b_n^{r'}$ is convergent. Then Σa_n^r is convergent. There are corresponding results for the convergence of infinite integrals (of both kinds).*

Note that this theorem is not a trivial consequence of Theorem 3 (for sums).

It is sufficient to consider the series form, and to prove: if Σa_n^r is divergent there exists a set of b's such that $\Sigma b_n^{r'}$ is convergent and $\Sigma a_n b_n$ divergent; or, writing $a_n^r = a_n$, $a_n b_n = t_n a_n$, to prove: given a divergent Σa_n we can find a set of t_n such that $\Sigma a_n t_n$ is divergent and $\Sigma a_n t_n^{r'}$ is convergent. For this it is sufficient to take $t_n = 1/s_n$, where

$$s_n = a_1 + a_2 + \dots + a_n.$$

[For $\Sigma \frac{a_n}{s_n}$ diverges with $\Pi\left(1-\frac{a_n}{s_n}\right) = \Pi\frac{s_{n-1}}{s_n}$,

and on the other hand

$$\sum^n \frac{a_n}{s_n^{r'}} = \sum^n \int_{s_{n-1}}^{s_n} \frac{dx}{s_n^{r'}} \leqslant \sum^n \int_{s_{n-1}}^{s_n} \frac{dx}{x^{r'}} = \int^{s_n} \frac{dx}{x^{r'}} < K.]$$

2.91. THEOREM (Young). *Let $\phi(x) \geqslant 0$, $\psi(y) \geqslant 0$, $\phi(0) = \psi(0) = 0$, and let $y = \phi(x)$, $x = \psi(y)$ be strictly increasing, continuous, and inverses of each other in $x \geqslant 0$, $y \geqslant 0$. Then, if $a \geqslant 0$, $b \geqslant 0$,*

$$ab \leqslant \int_0^a \phi(x)\,dx + \int_0^b \psi(y)\,dy, \tag{1}$$

and equality occurs if and only if $b = \phi(a)$.

† Note that $\operatorname{sgn} f = \operatorname{sgn}\,[f]_N$.

Either $a = \phi(\alpha) \leqslant b = \phi(\beta)$, or else $\beta' = \psi(b) \leqslant \alpha = \psi(a')$; say the former. Then $\beta \geqslant \alpha$.

$$\int_0^a \phi\, dx + \int_0^b \psi\, dy - ab = \int_0^a \phi(x)\, dx + \int_0^\beta x\, d\phi(x) - ab$$

$$= \beta\phi(\beta) - \int_a^\beta \phi(x)\, dx - ab$$

$$\geqslant \beta\phi(\beta) - (\beta - a)\,\phi(\beta) - ab = 0.$$

The inequality

$$ab \leqslant \frac{a^r}{r} + \frac{b^{r'}}{r'} \tag{2}$$

is the particular case $\phi(x) = x^{r-1}$. It generalizes to

$$abc \ldots \leqslant \Sigma \frac{a^r}{r} \quad \text{for} \quad \Sigma \frac{1}{r} = 1. \tag{3}$$

for more than two numbers $a, b, c, \ldots$. This, in turn, generalizes as follows.

2.92. THEOREM. *Let $\phi_1(x), \ldots \phi_m(x)$ be continuous, strictly increasing functions of x in $x \geqslant 0$, and $\phi_n(0) = 0$ $(n \leqslant m)$. Let $\Phi_n(x) = x\phi_n(x)$, and let $x = X_n(y)$ be the function inverse to $y = \Phi_n(x)$. Suppose now that the ϕ_n are connected by*

$$\prod_1^m X_n(y) = y \quad (y \geqslant 0), \tag{1}$$

or, more generally, by

$$\prod_1^m X_n(y) \leqslant y \quad (y \geqslant 0). \tag{1)'$$

Then for $a_n \geqslant 0$ $(n \leqslant m)$,

$$\prod_1^m a_n \leqslant \sum_1^m \int_0^{a_n} \phi_n(x)\, dx. \tag{2}$$

We may suppose that no a_n is zero. Consider

$$F(a_1, a_2, \ldots a_m) = \Sigma \int_0^{a_n} \phi_n(x)\, dx \quad (a_n \geqslant 0), \tag{3}$$

subject to

$$G = \prod^m a_n = t, \tag{4}$$

where t is a positive constant. F is evidently bounded below, and it tends to ∞ as any a_n tends to infinity. It follows (since F is continuous)

that F attains its absolute minimum for some set of a_n's, none of which is zero.

For this set we have, by a theorem of the differential calculus,

$$\frac{\partial F}{\partial a_n} - \lambda \frac{\partial G}{\partial a_n} = 0 \quad (n \leqslant m), \tag{5}$$

for *some* value of λ, independent of n. (5) gives

$$\Phi_n(a_n) = a_n \frac{\partial F}{\partial a_n} = \lambda a_n \frac{\partial G}{\partial a_n} = \lambda t,$$

$$a_n = X_n(\lambda t)\,; \tag{6}$$

and taking the product for $n \leqslant m$,

$$t = \Pi X_n(\lambda t) \tag{7}$$

$$\leqslant \lambda t, \text{ by } (1)'.$$

The right-hand side of (7) being a strictly increasing function of λ, the equation (7) has at most one solution for λ given t. Hence there is *exactly* one solution $\lambda \geqslant 1$, for which (6) gives a minimal set of a_n. For this set the (minimum) value F^* of F is

$$F^* = \sum_1^m \int_0^{X_n(\lambda t)} \phi_n(x)\, dx.$$

Transform the integral by $x = X_n(\xi)$, when $\phi_n = \Phi_n/x = \xi/X_n(\xi)$. We have

$$F^* = \Sigma \int_0^{\lambda t} \frac{\xi dX_n(\xi)}{X_n(\xi)} \geqslant \Sigma \int_0^{\lambda t} \Pi X_n(\xi) \frac{dX_n(\xi)}{X_n(\xi)} = \int_0^{\lambda t} d\Pi X_n(\xi) \tag{8}$$

$$= \Pi X_n(\lambda t) = t.$$

Hence for *any* set $a_1, \ldots a_n$,

$$F \geqslant F^* = t = \Pi a_n,$$

the desired result.

The relation (1) is the generalization of the relation between a pair of inverse functions ϕ, ψ. To see this, let $y = \phi(x)$, $x = \psi(y)$. Then

$$xy = \Phi(x) = \Psi(y)\,; \quad x = \Phi^{-1}(xy), \quad y = \Psi^{-1}(xy),$$

$$xy = \Phi^{-1}(xy)\, \Psi^{-1}(xy).$$

For a general function there is no extension, universally true, of (H) in *product* form.

3. THEOREM 5 (*the "selection principle"*). *Suppose that the numbers $\beta(n,m)$ exist for all integral values of n and m, and are bounded for each fixed m and varying n. Then there exists a strictly increasing sequence n_1, n_2, ... and a set of numbers β_1, β_2 ... such that for each value of m, $\beta(n_r, m)$ tends to β_m as $r \to \infty$.*

The numbers $\beta(n, 1)$ have at least one limit point β_1 say, and we can select a strictly increasing sequence S_1, or $n_{1,1}$, $n_{1,2}$, ..., such that $\beta(n_{1,r}, 1) \to \beta_1$. The numbers $\beta(n_{1,r}, 2)$ have a limit point, β_2 say, and we can find an (increasing) subsequence S_2 of S_1, or $n_{2,1}$, $n_{2,2}$, ..., for which $\beta(n_{2,r}, 2) \to \beta_2$. Further we may evidently suppose that $n_{2,2} > n_{1,1}$, for example by choosing $n_{2,1} = n_{1,2}$. The process can be continued indefinitely: there exists β_3 and S_3, or $n_{3,1}$, $n_{3,2}$, ..., a subsequence of S_2 with $n_{3,3} > n_{2,2}$ and so on. Consider now the increasing sequence S, or $n_{1,1}$, $n_{2,2}$, $n_{3,3}$, ... For each m we have $\beta(n,m) \to \beta_m$ as $n \to \infty$ through S_m; hence also $\beta(n,m) \to \beta_m$ as $n \to \infty$ through S, which is a subsequence of S_m.

If the variable m (but not n) in $\beta(n, m)$ is replaced by a continuous real or complex variable, the principle ceases to hold. We may, however, expect it to hold if $\beta(n, z)$ has sufficient continuity for its behaviour to be dominated by the behaviour of $\beta(n, z_m)$, where (z_m) is a set of points everywhere dense. We have, in fact, the following important and powerful principle.

COROLLARY. *Suppose that f_n is continuous in a bounded domain D of one or two (or any finite number of) dimensions, and that in any closed set D_1' interior to D the continuity is uniform in n. Then there exists a subsequence (n_r) and a continuous "limit-function" f such that $f_{n_r} \to f$ as $r \to \infty$, uniformly in any closed set interior to D.*

The uniformity of the continuity means that given ϵ there exists a $\delta = \delta(\epsilon, D_1')$ (independent of n) such that for all n and all z, z' of D_1 subject to $|z-z'| < \delta$ we have

$$(1) \qquad |f_n(z') - f_n(z)| < \epsilon.$$

Consider the points of D whose coordinates are rational. They can be arranged in a progression (z_m) in such a manner that for any z of D there exists a z_m within distance δ of it for which $m \leqslant M$, where

$$M = M(\delta) = M(\epsilon)$$

depends only on δ and D (for example, by arranging that the greatest denominator occurring in coordinates of z_m is monotonic increasing with m). The numbers $\beta(n, m) = f_n(z_m)$ are bounded for fixed m and varying

n. Hence there exists a sequence $n_1, n_2, \ldots$ and numbers $a_1, a_2, \ldots$ such that for each m

$$\phi_r(z_m) = f_{n_r}(z_m) \to a_m$$

as $r \to \infty$. [Note that the sequence (n_r) and the functions ϕ_r do not depend on D_1'.] Now let D_2' be any closed set interior to D, a the distance from D_2' to the frontier of D, and let D_1' (so far arbitrary) be the set of points of D whose distance from the frontier is not less than $\frac{1}{2}a$. Let δ be the number associated with D_1' and the inequality (1). We may allow δ to be diminished, and may therefore suppose that $\delta < \frac{1}{2}a$. If now z is any point of D_2' there exists a z_m within a distance δ of z, where $m \leqslant M(\delta)$, and z and z_m belong to D_1'. Now, given *any* m, there exists a $\nu(\epsilon, m)$ such that

$$|\phi_r(z_m) - \phi_s(z_m)| < \epsilon \quad (r, s \geqslant \nu);$$

hence, if

$$\operatorname*{Max}_{m \leqslant M} \nu(\epsilon, m) = N = N(\epsilon, \delta) = N(\epsilon, D_2'),$$

we have

$$|\phi_r(z_m) - \phi_s(z_m)| < \epsilon \quad (r, s \geqslant N,\ m \leqslant M). \tag{2}$$

In the inequality

$$|\phi_r(z) - \phi_s(z)| \leqslant |\phi_r(z) - \phi_r(z_m)| + |\phi_s(z) - \phi_s(z_m)| + |\phi_r(z_m) - \phi_s(z_m)|$$

each term on the right is less than ϵ (if $r, s \geqslant N$), the first two in virtue of $|z - z_m| < \delta$ and (1), the last in virtue of (2). Thus

$$|\phi_r(z) - \phi_s(z)| \leqslant 3\epsilon$$

for $r, s \geqslant N(\epsilon)$, and all z of D_2'. It follows that $\phi_r(z)$ tends to a limit function $f(z)$ uniformly in D_2'. Since D_2' is arbitrary, $f(z)$ cannot depend on it; finally f is continuous at any point of D, as the uniform limit of a continuous function ϕ_r. This completes the proof.

4. *Theory of functions of a real variable.*

4.1. In this section we set out those parts of the theory of real functions which we require later. We sometimes carry our developments beyond what is strictly necessary, but have not tried to be exhaustive. The extent of knowledge required is nothing like so great as is sometimes supposed. There are three principles, roughly expressible in the following terms: Every (measurable) set is nearly a finite sum of intervals; every function (of class L^λ) is nearly continuous; every convergent sequence of functions is nearly uniformly convergent. Most of the results of the present section are fairly intuitive applications

of these ideas, and the student armed with them should be equal to most occasions when real variable theory is called for. If one of the principles would be the obvious means to settle a problem if it were "quite" true, it is natural to ask if the "nearly" is near enough, and for a problem that is actually soluble it generally is.

When our results are capable of extension to functions that may take complex values—and they generally are—the extended form is either deducible trivially from the real one, or else the *proof* for the real case applies with trivial modifications. We shall therefore take such extensions for granted, giving only the proof in the real case, and sometimes only the *enunciation* in the real case, when either procedure suits our convenience.

4 . 21. THEOREM 6†. *Suppose that $f(\theta)$ is of class L^λ in E_0, and let δ, ϵ be given. Then there exist (a) a continuous ϕ, (b) a stretchwise constant‡ ("step-function") ϕ, with the further properties:*

(1) $|f-\phi| < \epsilon$, *except in a set e of measure less than δ,*

(2) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f-\phi|^\lambda \, d\theta \leqslant \epsilon^\lambda,$$

(3) $$\phi(\pi) = \phi(-\pi).$$

If further f is bounded on one or both sides, such a can be found with the same bound or bounds.

Theorem 2 shows that if, given f, we can find, first an $f^{(1)}$ of a certain type such that $M_\lambda(f-f^{(1)})$ is arbitrarily small, then, for fixed $f^{(1)}$, an $f^{(2)}$ of another type such that $M_\lambda(f^{(1)}-f^{(2)})$ is arbitrarily small, and so on to $f^{(r)}$; then, given f, we can find a function of the last type, $f^{(r)}$, such that $M_\lambda(f-f^{(r)})$ is arbitrarily small. We shall use this argument frequently, and we have here its first occasion. A precisely similar principle is available for a result of the type (1).

Let $f_n = [f]_n$ and let e_n be the set in which $f_n \neq f$. Then $n = |f_n| \leqslant |f|$ in e_n,

$$n^\lambda e_n \leqslant \int_{e_n} |f|^\lambda \, d\theta \leqslant \int_{E_0},$$

and so $e_n \to 0$ as $n \to \infty$, and

$$\int_{E_0} |f-f_n|^\lambda \, d\theta \leqslant \int_{e_n} |2f|^\lambda \, d\theta \to 0.$$

† This is the second principle of § 4 . 1.

‡ That is to say: E_0 can be divided into a finite number of intervals, in each of which ϕ is a constant.

By the principle explained above we may, in proving (1) and (2), now suppose that f is bounded. We may, for similar reasons, reduce the proof further to the case when f is a function which takes only a finite number of values $a_1, a_2, \ldots, a_N$. [If f^* is the integral multiple of ϵ' nearest to f we have $|f^*-f| < \epsilon'$.] Let now e_r be the set in which $f = a_r$. Then $e_r = e'_r + e''_r - e'''_r$, where e''_r and e'''_r have arbitrarily small measure and e'_r is a finite sum of intervals†; moreover, the dissections can clearly be made in such a way that $e'_1, e'_2, \ldots e'_N$ are mutually exclusive and together compose E_0. Then if f^* is the function which is a_r in e'_r $(r = 1, 2, \ldots, N)$, $M_\lambda(f^*-f)$ is arbitrarily small, and f^* is a stretchwise constant function ϕ with the properties (1) and (2). By bridging the gaps in the graph of this ϕ by sufficiently steep lines [thereby disturbing the values of the function in a set of arbitrarily small measure, and disturbing $M_\lambda(f-\phi)$ arbitrarily little] we can convert ϕ into a continuous function, which, by one more such adjustment, can be made also to satisfy (3). The stretchwise constant ϕ itself can be made to satisfy (3) by an arbitrarily slight adjustment. Thus we have proved the existence of ϕ's of types (a) and (b) which satisfy (1), (2), and (3). To prove the last part of the theorem we have only to observe that if, *e.g.*, $\alpha \leqslant f \leqslant \beta$ and ϕ satisfies (1), (2), (3), then ϕ^*, defined to be ϕ, α, or β, according as $\alpha \leqslant \phi \leqslant \beta$, $\phi < \alpha$, or $\phi > \beta$, is of the same type [(a) or (b)] as ϕ, and satisfies *a fortiori* the conditions (1), (2), (3).

A less elementary but shorter and more elegant proof proceeds as follows. [It appeals to theorems proved a little later, but they are without logical priority to Theorem 6.]

Let

$$f^* = |f|^\lambda \operatorname{sgn} f, \quad \phi^*(\theta) = 2n\int_{-1/n}^{1/n} f^*(\theta+t)\,dt, \quad \phi = |\phi^*|^{1/\lambda} \operatorname{sgn} \phi^*.$$

ϕ is continuous [and satisfies (3)], and it is easily verified that bounds of f are also bounds of ϕ. Since f^* is of class L and is therefore almost always the derivative of its integral, the functions

$$\pm n\int_0^{l} f^*(\theta+t)\,dt,$$

and therefore also $\phi^*(\theta)$, converge almost always to $f^*(\theta)$ as $n \to \infty$. Hence also ($\operatorname{sgn} \phi^* \to \operatorname{sgn} f^*$ and) $\phi \to f$ p.p. Also

$$\int_{E_0} |\phi|^\lambda d\theta = \int_{E_0} |\phi^*|\, d\theta \leqslant 2n\int_{-1/n}^{1/n} dt \int_{E_0} |f^*(\theta+t)|\, dt = \int_{E_0} |f(\theta)|^\lambda d\theta.$$

† This (for an arbitrary set) is the first principle of § 4.1.

By Theorem 13 (2) holds when n is large, and (1) is an immediate consequence of Theorem 9. This disposes of case (a); the transition to a step-function is, of course, easy.

4.22. THEOREM 7. *Let $f(\theta)$ be periodic and of class L^λ. Then*

$$F(\theta) = \int_{-\pi}^{\pi} |f(t+\theta) - f(t)|^\lambda \, dt$$

is periodic and continuous. In particular $F(\theta) \to 0$ as $\theta \to 0$.

Given ϵ we can find a continuous and periodic ϕ for which

$$\int |f - \phi|^\lambda \, d\theta < \epsilon.$$

Let $$\Phi(\theta) = \int_{-\pi}^{\pi} |\phi(t+\theta) - \phi(t)|^\lambda \, d\theta.$$ By Theorem 2 (14)

$$|\{F(\theta)\}^a - \{\Phi(\theta)\}^a|$$
$$\leqslant \left(\int_{-\pi}^{\pi} |f(t+\theta) - \phi(t+\theta)|^\lambda \, dt\right)^a + \left(\int_{-\pi}^{\pi} |f(t) - \phi(t)|^\lambda \, dt\right)^a < 2\epsilon^a.$$

Since $\Phi(\theta)$ is continuous it follows that $\{F(\theta)\}^a$ and $\{F(\theta')\}^a$ differ by at most $3\epsilon^a$ provided $\theta' - \theta < \delta(\epsilon)$. It follows that $\{F(\theta)\}^a$ and $F(\theta)$ are continuous.

COROLLARY. *Let f and g be periodic, and f of class L^r and g of class $L^{r'}$, or f of class L and g bounded. Then*

$$H(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta + t) \, g(t) \, dt$$

is a continuous function of θ.

We have in the first case (writing $t - \theta$ for t)

$$|\Delta H| = |H(\theta + \delta) - H(\theta)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \{f(t+\delta) - f(t)\} \, g(t - \theta) \, dt \right|$$

(4) $$\leqslant \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t+\delta) - f(t)|^r \, dt \right\}^{1/r} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(t - \theta)|^{r'} \, dt \right\}^{1/r'}$$

The first factor is small with δ, by the main theorem, the second is $M_{r'}(g)$† and so finite. Thus $\Delta H \to 0$ with δ.

† This disappearance of an apparent parameter (here θ) will become very familiar.

In the second case the right-hand side of (4) is replaced by

$$\frac{K}{2\pi}\int_{-\pi}^{\pi}|f(t+\delta)-f(t)|\,dt,$$

where K is a bound of $|g|$, and the conclusion follows as before.

4.23. THEOREM 8 (Riemann-Lebesgue). *If E is any set (of finite or infinite measure) and $\int_E |f|\,d\theta$ exists, then*

$$\lim_{t\to\infty}\int_E f(\theta)\,e^{t\theta i}\,d\theta = 0.$$

Let $[E]_N$ be the set of those points of E for which $|\theta| \leqslant N$. Then

$$\left|\int_{E-[E]_N} f(\theta)\,e^{t\theta i}\,d\theta\right| \leqslant \int_{E-[E]_N}|f|\,d\theta \to 0$$

as $N\to\infty$. It is therefore enough to prove the theorem for a bounded set, or for E_0 and a periodic f.

(i) Writing η for πt^{-1},

$$F(t) = \int_{-\pi}^{\pi} f(\theta)\,e^{t\theta i}\,d\theta = -\int_{-\pi+\eta}^{\pi+\eta} f(\theta-\eta)\,e^{t\theta i}\,d\theta,$$

and, adding and taking half the sum,

$$F(t) = \tfrac{1}{2}\int_{-\pi}^{\pi}\{f(\theta)-f(\theta-\eta)\}\,e^{t\theta i}\,d\theta$$

$$+\tfrac{1}{2}\int_{-\pi}^{-\pi+\eta} f(\theta-\eta)\,e^{t\theta i}\,d\theta - \tfrac{1}{2}\int_{\pi}^{\pi+\eta} f(\theta-\eta)\,e^{t\theta i}\,d\theta.$$

$$|F(t)| \leqslant \tfrac{1}{2}\int_{E_0}|f(\theta-\eta)-f(\theta)|\,d\theta + \tfrac{1}{2}\int_{e_1}|f(\theta)|\,d\theta + \tfrac{1}{2}\int_{e_2}|f(\theta)|\,d\theta,$$

when e_1 and e_2 have measure η. Each term on the right tends to zero as $t\to\infty$ (or $\eta\to 0$), the first term in virtue of Theorem 7.

(ii) It is easy to prove directly from Theorem 6 that it is enough to prove Theorem 8 when f is a step function and E an arbitrary finite interval. This reduces to proving it when f is a constant, and this is immediate, by direct integration.

4.31. THEOREM 9 (Egoroff)†. *Suppose that as $n\to\infty$ $f_n(\theta)\to f(\theta)$* p.p. *in $E \subset E_c$, and that f is finite* p.p. *Then there exists a set H,*

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† This is the third principle of § 4.1.

of arbitrarily small measure, such that $f_n \to f$ *uniformly in* $E-H$. *The result holds also if* f *is (say)* $+\infty$ p.p.

The last part follows by a trivial transformation from the first, which we proceed to consider.

Let $\phi_n(x)$ be the upper bound of $|f-f_m|$ for $m \geqslant n$: ϕ_n is a decreasing function of n for each x. Let $E_n(\epsilon)$ be the set in which $\phi_n \geqslant \epsilon$. For fixed ϵ $E_n(\epsilon)$ is a decreasing set, and its limit as $n \to \infty$ is included in the null-set in which f_n does not tend to f. Let now ϵ_n tend to 0 as $n \to \infty$ and let Σa_n be a convergent series of positive terms.

Since $mE_n(\epsilon) \to 0$ there exists an n_r such that $mE_{n_r}(\epsilon_r) < a_r$. The measure of the set

$$H = \sum_r^\infty E_{n_r}(\epsilon_r)$$

does not exceed $\sum_r^\infty a_r$, which is less than η by choice of r. Consider now any point θ of $E-H$, and let $n > n_{r+s}$, where s will be chosen presently. We have

$$|f_n(\theta)-f(\theta)| \leqslant \phi_n(\theta) \leqslant \phi_{n_{r+s}}(\theta) \leqslant \epsilon_{r+s},$$

since θ belongs to the complementary set of $E_{n_{r+s}}(\epsilon_{r+s})$. Since $\epsilon_{r+s} < \epsilon$ for $s > s_0$, where s_0 is independent of θ, it follows that $f_n \to f$ uniformly in $E-H$.

4.32. THEOREM 10. *If* $f_n \to f$ p.p. *in* $E \subset E_0$, *then*

$$\int_E f_n d\theta \to \int_E f d\theta,$$

provided f_n *is uniformly bounded*†, *or the product of a uniformly bounded function and a function of class L independent of n, or, more generally, provided* f_n *has a majorant of class L and independent of n.*

For if F is the majorant of the f_n (and therefore also a majorant of f), and H is the set of Theorem 9, we have

$$\overline{\lim_{n\to\infty}} \int_E |f_n - f| \, d\theta \leqslant \overline{\lim} \int_{E-H} + \overline{\lim} \int_H = \overline{\lim} \int_H \leqslant \overline{\lim} \int_H 2F d\theta,$$

and this is arbitrarily small with the measure of H.

A similar result holds, of course, for a function depending on a continuous parameter instead of on n, and the same thing applies to many of the theorems that follow. We shall state explicitly only the results involving the parameter n.

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† In these circumstances we say that f_n converges to f boundedly.

4.33. THEOREM 11. (i) *Suppose* $a_r(n) \geqslant 0$, $a_r(n) \to a_r$ *as* $n \to \infty$. *Then*

(1) $$\sum_1^\infty a_r \leqslant \varliminf_{n\to\infty} \sum_1^\infty a_r(n).$$

(ii) *Suppose* $f_n(\theta) \geqslant 0$ *and* $f_n \to f$ p.p. *in* $E \subset E_0$. *Then*

(2) $$\int_E f\,d\theta \leqslant \varliminf_{n\to\infty} \int_E f_n\,d\theta.$$

To prove (i) we need only observe that for every N

$$\sum^N a_r = \lim_{n\to\infty} \sum^N a_r(n) \leqslant \varliminf \sum^\infty a_r(n).$$

For (ii), the analogue of (i) for integrals, we have $[f_n]_N \to [f]_N$ p.p., *and boundedly*. Hence, by Theorem 10,

$$\int_E [f]_N\,d\theta = \lim \int_E [f_n]_N\,d\theta \leqslant \varliminf \int_E f_n\,d\theta,$$

since $f_n \geqslant 0$ and so $[f_n]_N \leqslant f_n$. The limit of the left side as $N \to \infty$ is $\int_E f\,d\theta$ *by definition*, and the desired result follows.

The proofs are valid whether the right-hand sides of (1) and (2) are finite or $+\infty$. When $\int f d\theta$ is finite (ii) is an immediate corollary of Theorem 9. Thus

$$\varliminf \int_E (f_n - f)\,d\theta \geqslant \varliminf \int_{E-H} (f_n - f)\,d\theta + \varliminf \int_H f_n\,d\theta - \int_H f\,d\theta,$$

and of the terms on the right the first is zero, the second non-negative, and the third arbitrarily small with mH, whence $\varliminf \int_E \geqslant 0$.

COROLLARY. *If* f_n *is monotonic increasing (decreasing) in* n *for each* θ *of* E, *so that* f_n *converges to some* f *in* E, *and if* $\int_E f_n d\theta > -\infty$ $(< +\infty)$ *for some* $n = \nu$, *then*

$$\lim \int_E f_n\,d\theta = \int_E f\,d\theta.$$

For supposing f_n increasing, $f_n - f_\nu$ is non-negative for $n > \nu$ on the one hand, and not greater than $f - f_\nu$ on the other, so that

$$\int_E (f - f_\nu)\,d\theta \leqslant \varliminf \int_E (f_n - f_\nu)\,d\theta \leqslant \varlimsup \leqslant \int_E (f - f_\nu)\,d\theta.$$

Suppressing the various $-f_\nu$ we have

$$\lim \int_E f_n d\theta = \int_E f d\theta.$$

4.34. THEOREM 12. *Suppose that $a_r(n) \to a_r$, for each r, as $n \to \infty$, and that $\sum_1^\infty |a_r|^\lambda$ is convergent. Then in order that*

(1) $$\lim_{n\to\infty} \sum_{r=1}^\infty |a_r(n) - a_r|^\lambda = 0,$$

it is sufficient, and also necessary, that

(2) $$\Sigma |a_r(n)|^\lambda \to \Sigma |a_r|^\lambda.$$

The necessity is given by Theorem 2 (18).† For sufficiency we assume the truth of (2) and have to prove (1). We have

$$\lim_{n\to\infty} \sum_N^\infty |a_r(n)|^\lambda = \lim \sum_1^\infty |a_r(n)|^\lambda - \lim \sum_1^{N-1} |a_r(n)|^\lambda = \sum_1^\infty |a_r|^\lambda - \sum_1^{N-1} |a_r| = \epsilon_N.$$

Hence, by Theorem 2 (16),

$$\overline{\lim} \sum_N^\infty |a_r(n) - a_r|^\lambda \leqslant \overline{\lim}\, A_\lambda \sum_N^\infty \left(|a_r(n)|^\lambda + |a_r|^\lambda\right) \leqslant 2A_\lambda \epsilon_N.$$

Hence finally

$$\overline{\lim} \sum_1^\infty |a_r(n) - a_r|^\lambda = \overline{\lim} \sum_N^\infty \leqslant 2A_\lambda \epsilon_N,$$

and the left-hand side must be zero.

The same arguments establish the following

COROLLARY. *Let $f_r(\theta)$ $(r = 1, 2, \ldots)$ be a set of non-negative functions each continuous at $\theta = \theta_0$. If now*

$$F(\theta) = \Sigma f_r(\theta)$$

is continuous at θ_0, then

$$\lim_{\theta\to\theta} \Sigma |f_r(\theta) - f_r(\theta_0)| = 0,$$

and $\Sigma f_r(\theta)$ is uniformly convergent in some neighbourhood of $\theta = \theta_0$.

The first part is a particular case (with $\lambda = 1$) of the theorem (for a continuous variable θ). For the second ("Dini's theorem") we have

$$\sum_N^\infty f_r(\theta) \leqslant \sum_N^\infty f_r(\theta_0) + \sum_0^\infty |f_r(\theta) - f_r(\theta_0)|.$$

† Indeed the hypothesis $a_r(n) \to a_r$ is here unnecessary, being a *consequence* of (1). But this no longer applies in Theorem 13, the corresponding result for integrals.

The first term on the right is independent of θ, the second is less than ϵ provided that $|\theta-\theta_0| < \delta(\epsilon)$, by the first part, and the result follows at once.

4.35. The result for integrals corresponding to Theorem 12 is as follows (we have developed certain details):

THEOREM 13. *Suppose that* $f_n \to f$ p.p. *in* E_0 *and* $\int_{E_0} |f|^\lambda d\theta < \infty$. *Then in order that*

(1) $$\int_{E_0} |f_n - f|^\lambda d\theta \to 0,$$

it is sufficient, and also necessary, that

(2) $$\int_{E_0} |f_n|^\lambda d\theta \to \int_{E_0} |f|^\lambda d\theta,$$

and in particular it is sufficient that

(3) $$\int_{E_0} |f_n|^\lambda d\theta \leqslant \int_{E_0} |f|^\lambda d\theta.$$

When (2) *holds we have also*

(4) $$\int_E |f_n|^\lambda d\theta \to \int_E |f|^\lambda d\theta \quad (E \subset E_0).$$

In particular, when $\lambda = 1$, (2) *is a sufficient condition for*

(5) $$\int_E f_n d\theta \to \int_E f d\theta \quad (E \subset E_0),$$

but it is not necessary.

Finally, if either (and therefore both) of (1) *and* (2) *holds for a particular* λ, *they hold for all smaller* λ.

We follow the proof of Theorem 12 so far as is possible. It is instructive to note what elementary device in Theorem 12 corresponds to the use of Theorem 9 in Theorem 13.

In the first place, by Theorem 2 (18), (1) implies (2) (and, indeed, without the hypothesis $f_n \to f$ p.p.). Suppose now that (2) holds. It is easy to see that (2) *must hold also when* E_0 *is replaced by any* $E \subset E_0$. For writing

$$J_n(E) = \int_E |f_n|^\lambda d\theta, \quad J(E) = \int_E |f|^\lambda d\theta,$$

we have, by Theorem 11, $\underline{\lim}\, J_n(E) \geqslant J(E)$. If now $\overline{\lim}\, J_n(E) > J(E)$, we have, by combination with $\underline{\lim}\, J_n(CE) \geqslant J(CE)$, $\overline{\lim}\, J_n(E_0) > J(E_0)$, contrary to hypothesis. Thus $\overline{\lim}\, J_n(E) \leqslant J(E)$ and $\lim J_n(E) = J(E)$.

We can now deduce (1) from (2). By Theorem (9) $|f_n-f|\to 0$ uniformly in E_0-H, where mH is arbitrarily small. Then

$$\int_{E_0-H}|f_n-f|^\lambda\,d\theta\to 0.$$

Also
$$\int_H|f_n-f|^\lambda d\theta\leqslant A_\lambda\int_H|f_n|^\lambda\,d\theta+A_\lambda\int_H|f|^\lambda\,d\theta.$$

The right-hand side tends to $2A_\lambda\int_H|f|^\lambda\,d\theta$, by what we have just proved, and this is arbitrarily small with mH. It follows that

$$\overline{\lim}\int_{E_0}|f_n-f|^\lambda\,d\theta=\overline{\lim}\int_H$$

is arbitrarily small and therefore zero.

Since (3) implies (2), by Theorem 12, it is sufficient for the truth of (1).

The result (5) is, of course, included in (4). To see that, however, (2) (with $\lambda=1$) is not necessary for the truth of (5), consider the example

$$f_n=0\ \ (\theta=0),\quad f_n=n\ \ (0<|\theta|\leqslant n^{-1}),\quad f_n=-n\ \ (n^{-1}<|\theta|\leqslant 2n^{-1}),$$
$$f_n=0\ \ (|\theta|>2n^{-1}).$$

Here $f(\theta)=0$, but
$$\int_{E_0}|f_n|\,d\theta=2.$$

To prove the last part of the theorem we have only to observe that if $\int_{E_0}|f_n-f|^\lambda\,d\theta$ tends to zero for a particular λ it does so also for any smaller λ, by Theorem 1 (10).

4.41. Let $(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_N,\beta_N)$ be any finite set of non-overlapping, but possibly abutting, intervals contained in E_0. A function f is said to be of bounded variation (b.v.) in E_0 if a K exists such that

$$t=\sum_{m=1}^{N}|f(\beta_m)-f(\alpha_m)|\leqslant K$$

for all such sets of intervals. If, further, given ϵ, there exists a $\delta=\delta(\epsilon)$ such that $t<\epsilon$ for every such set of intervals whose total measure does not exceed δ, the function f is said to be absolutely continuous (a.c.).

If f depends on a parameter n and if numbers corresponding to K, δ above exist which are independent of n, then f_n is said to be uniformly b.v. or uniformly a.c. (u.b.v. or u.a.c.).

In what follows we use e to denote the set of points belonging to any finite set of non-overlapping intervals contained in E_0, and (α_n, β_n) $(n = 1, 2, \ldots, N)$ for the intervals themselves. We shall sometimes use the symbol e to denote also the mode of dissection of the set e into intervals. [Note that distinct modes of division e may correspond to the same set of points e.]

Any measurable set E is of the form $e+E_1-E_2$, where mE_1, mE_2 are arbitrarily small. Let now mE_1, mE_2, and Max $|\beta_m-\alpha_m|$ tend to zero in any manner. Then it is known that if f is a.c. the number t tends to a unique finite limit $T(f, E)$, depending only on f and E, and additive in E. T is called the total variation (t.v.) of f in E.

For a b.v. function and an *interval* $\alpha\beta$, the t.v. is defined to be the upper bound of t for all modes of division of $\alpha\beta$ with an e. The t.v. for an e is defined as the sum of the t.v.'s for the intervals composing e (this depends on e only *qua* set of points). When f is a.c. and E is an e there is evidently consistency with the definition of $T(f, E)$. We shall never have occasion to apply the conception of the t.v. of a b.v. function in an *arbitrary* E, and since it would require a long explanation we shall ignore it.

The further results contained in Theorem 14 and 15 are also known.

THEOREM 14. (i) *A* b.v. *function is the difference of two positive and increasing functions, which may be supposed continuous if f is continuous.* (ii) *A* b.v. *function f has* p.p. *a finite differential coefficient f', and f' belongs to the class* L. (iii) *If f is non-decreasing and continuous, then*

$$\int_\alpha^\theta f'(\theta)\, d\theta \leqslant f(\theta)-f(\alpha) \qquad (\theta \geqslant \alpha).$$

THEOREM 15. *A function f* a.c. *in E_0 is of the form*

$$\int_0^\theta g(\theta)\, d\theta + c,$$

where g is of class L *in E_0 and c is a constant. Further*

$$T(f, E) = \int_E |g|\, d\theta,$$

so that T tends uniformly to 0 with mE (i.e. uniformly in E)†. *Conversely the integral of a function of class* L *is* a.c.

† This is true by definition when E is an e.

4 . 42. THEOREM 16. *Suppose that f has the property*

(1) $$\sum_{n=1}^{N} \frac{|f(\beta_n)-f(\alpha_n)|^r}{|\beta_n-\alpha_n|^{r-1}} \leqslant G$$

for every set e, where $r>1$ and G are constants. Then there exists a function g of class L^r such that

(2) $$f(\theta) = \int_0^{\theta} g\,d\theta + c$$

and

(3) $$\int_{E_0} |g|^r d\theta \leqslant G.$$

Conversely (2) *and* (3) *together imply* (1).

The converse is immediate, since

$$\frac{|f(\beta_n)-f(\alpha_n)|^r}{|\beta_n-\alpha_n|^{r-1}} = |\beta_n-\alpha_n| \left| \frac{1}{\beta_n-\alpha_n} \int_{\alpha_n}^{\beta_n} g\,d\theta \right|^r$$

$$\leqslant |\beta_n-\alpha_n| \frac{1}{|\beta_n-\alpha_n|} \int_{\alpha_n}^{\beta_n} |g|^r d\theta,$$

by Theorem 1 (10), whence the sum taken over the left-hand side does not exceed $\int_{E_0} |g|^r d\theta \leqslant G.$

Suppose, then, that (1) holds. We have first, by Theorem 1 (4), (8),

$$\sum_1^N |f(\beta_n)-f(\alpha_n)| = \Sigma \left|\frac{\Delta f}{\Delta\theta}\right| \Delta\theta \leqslant \left(\Sigma \left|\frac{\Delta f}{\Delta\theta}\right|^r \Delta\theta\right)^{1/r} (\Sigma\Delta\theta)^{1/r'} \leqslant G^{1/r}(me)^{1/r'}.$$

Since this tends to 0 with me f is a.c., and so, by Theorem 15, is of the form (2). Consider now a sequence of divisions e_n of E_0, e_n being, say, $(\theta_0^{(n)}, \theta_1^{(n)}, \ldots, \theta_{N_n}^{(n)})$, in which $\underset{(m \leqslant N_n)}{\operatorname{Max}} |\theta_{m+1}^{(n)}-\theta_m^{(n)}| \to 0$. We define a $g_n(\theta)$ for all θ of E_0 by

$$g_n(\theta) = \frac{f(\theta_{m+1}^{(n)})-f(\theta_m^{(n)})}{\theta_{m+1}^{(n)}-\theta_m^{(n)}} \quad (\theta_m^{(n)} \leqslant \theta \leqslant \theta_{m+1}^{(n)}).$$

Then if e is taken to be e_n (1) is identical with the inequality

(4) $$\int_{E_0} |g_n(\theta)|^r d\theta \leqslant G.$$

Now g_n is of the form

$$\int_{\theta-\epsilon_n}^{\theta+\epsilon_n'} g(t)\,dt \Big/ (\epsilon_n+\epsilon_n').$$

For a θ at which $g(\theta)$ is the differential coefficient of its integral the numerator is

$$\epsilon'_n\{g(\theta+o(1)\}+\epsilon_n\{g(\theta)+o(1)\}\ ;$$

hence $g_n(\theta)\to g(\theta)$ except for a null-set in θ. Hence, by Theorem 11,

$$\int_{E_0}|g|^r\,d\theta\leqslant\varliminf_{n\to\infty}\int_{E_0}|g_n|^r\,d\theta\leqslant G.$$

4.51. THEOREM 17. *If* f_n *is* u.a.c. *then*

$$f_n=\int_0^\theta g_n\,d\theta+c_n,$$

$_n$ *is of class* L *and* c_n *is a constant depending on* n†.
ϵ, *there exists* $\delta=\delta(\epsilon)$, *independent of* n, *such that, if* E *is any set* (*not necessarily independent of* n) *for which* $mE\leqslant\delta$, *then*

$$T(f_n,E)=\int_E|g_n|\,d\theta<\epsilon.$$

For (for fixed n) $T(f_n,E)$ differs arbitrarily little (see § 4.41) from a finite sum t taken for a set e of measure arbitrarily near mE.

4.52. THEOREM 18. *Suppose that* $f_n\to f$ p.p. *in* E_0. *Then in order that both* f *shall be of class* L *and* $\int_{E_0}|f_n-f|\,d\theta$ *shall tend to* 0 *it is necessary and sufficient that* $\int_0^\theta|f_n|\,d\theta$ *is* u.a.c., *and, again, it is necessary and sufficient that* $\int_0^\theta f_n\,d\theta$ *is* u.a.c. (*Hence also the last two conditions are equivalent.*)

The equivalence of the u.a.c. of $\int|f_n|\,d\theta$ and $\int f_n\,d\theta$ follows from Theorem 17 and we need only consider the first condition.

If $\int_0^\theta|f_n|\,d\theta$ is u.a.c., we have $\int_E|f_n|\,d\theta<1$ whenever $mE<\mu$, say. Then E_0 can be divided into $[2\pi/\mu]+1$ intervals e (at most) for each of which $\int_e|f_n|\,d\theta<1$; hence $\int_{E_0}|f_n|\,d\theta$ is bounded, and, by Theorem 11, f is of class L. We may therefore suppose, in *both* the necessary and the sufficient cases of our theorem, that f_n and f are of class L, and incidentally finite p.p. Then there exists a set H, with $mH<\delta$, such that

† Note that f_n need not be uniformly bounded: consider, *e.g.*, the case $f_n=c_n=n$.

$f_n - f \to 0$ uniformly in $E_0 - H$. We have, therefore, on the one hand

$$\int_{E_0} |f_n - f|\, d\theta \leqslant \int_{E_0 - H} |f_n - f|\, d\theta + \int_H |f|\, d\theta + \int_H |f_n|\, d\theta,$$

$$(1) \qquad \leqslant o(1) + \epsilon(\delta) + \int_H |f_n|,$$

and on the other

$$(2) \quad \int_E |f_n|\, d\theta \leqslant \int_E |f_n - f|\, d\theta + \int_E |f|\, d\theta \leqslant \int_{E_0} |f_n - f|\, d\theta + \epsilon_1(\delta) \quad (mE < \delta),$$

where $\epsilon(\delta)$ and $\epsilon_1(\delta)$ are independent of n, H, and E, and tend to 0 with δ. If now $\int |f_n|\, d\theta$ is u.a.c., the last term on the right-hand side of (1) is less than $\epsilon_2(\delta)$, and it follows that $\int_{E_0} |f_n - f|\, d\theta \to 0$. Conversely, if $\int |f_n|\, d\theta$ is not u.a.c., there exists an $\alpha > 0$ and a sequence $E_1, E_2, \ldots$ for which $mE_n \to 0$ and $\int_{E_n} |f_n|\, d\theta > \alpha$. If now we take $E = E_n$ in (2) and make $n \to \infty$, we have

$$\varliminf \int_{E_0} |f_n - f|\, d\theta \geqslant \alpha.$$

It follows that, if the integral last written tends to 0, $\int |f_n|\, d\theta$ must be u.a.c.

4.53. THEOREM 19. *Suppose that $f_n \to f$* p.p. *in E_0. Then in order that f shall be of class L and $\int_{E_0} |f_n - f|\, d\theta \to 0$ (or in order that $\int_0^\theta f_n\, d\theta$ shall be* u.a.c.) *it is sufficient, and also necessary, that a function $\Phi(t)$ exists such that $\phi(t) = \Phi(t)/t$ increases to $+\infty$ with t and*

$$(1) \qquad \int_{E_0} \Phi(|f_n|)\, d\theta < K$$

for all n.

Suppose that (1) holds. Given any set E let $\mu = mE$, and let E_1 be the sub-set of E in which $|f_n| \leqslant \mu^{-\frac{1}{2}}$, $E_2 = E - E_1$. Then

$$\int_{E_1} |f_n|\, d\theta \leqslant \int_{E_1} \mu^{-\frac{1}{2}}\, d\theta = \mu^{\frac{1}{2}} = \epsilon_1(\mu),$$

and

$$\int_{E_2} |f_n|\, d\theta \leqslant \{\phi(\mu^{-\frac{1}{2}})\}^{-1} \int_{E_2} |f_n|\, \phi(|f_n|)\, d\theta \leqslant K/\phi(\mu^{-\frac{1}{2}}) = \epsilon_2(\mu),$$

$$\int_E |f_n|\, d\theta < \epsilon(\mu),$$

so that $\int^{\theta} f_n d\theta$ is u.a.c. The sufficiency part of the theorem is therefore established (in virtue of Theorem 18).

In proving the necessity part we may suppose that $\int f_n d\theta$ is u.a.c., and also that f_n (and so f) is non-negative (Theorem 18), and we have to establish the existence of a Φ giving (1). As in § 4 . 52 we have $\int_{E_0} f_n\, d\theta < K'$. Let Σa_n be a convergent series of positive terms, ϵ_n a positive function of m tending to 0 as $m \to \infty$. Let T_m be a positive increasing function and $E_{m,n}$ the set in which $f_n > T_m$. Then

$$mE_{m,n} \leqslant \frac{1}{T_m}\int_{E_{m,n}} f_n\, d\theta \leqslant \frac{1}{T_m} \operatorname*{Max}_{(n)} \int_{E_0} f_n\, d\theta < \frac{K'}{T_m},$$

In virtue of the u.a.c. it is therefore possible (as a little reflection shows) to choose the function T_m in such a manner that

$$\int_{E_{m,n}} f_n\, d\theta \leqslant \epsilon_m a_m \quad (\text{all } n). \tag{2}$$

Let us now define $\Phi(t) = t\phi(t)$ where

$$\phi(t) = \epsilon_m^{-1} \quad (T_m \leqslant t < T_{m+1}).$$

Then

$$\int_{E_0} \Phi(f_n)\, d\theta = \left(\int_{E_0 - E_{1,n}} + \sum_{m=1}^{\infty} \int_{E_{m,n} - E_{m+1,n}}\right) f_n \phi(f_n)\, d\theta$$

$$\leqslant 2\pi T_1 + \sum_{m=1}^{\infty} \epsilon_m^{-1} \int_{E_{m,n}} f_n d\theta \leqslant 2\pi T_1 + \sum_1^{\infty} a_m = K.$$

COROLLARY 1. *Suppose that* $f_n \to f$ p.p. *in* E_0, $r > 1$, *and* $\int_{E_0} |f_n|^r d\theta \leqslant G$. *Then* $\int_{E_0} |f|^r d\theta \leqslant G$, $\int_{E_0} |f_n - f|\, d\theta \to 0$ *and (a fortiori)* $\int_E f_n\, d\theta \to \int_E f d\theta$ *for any fixed* $E \subset E_0$, *and* $\int_0^{\theta} f_n\, d\theta$ *is* u.a.c. *Finally we have* $\int_{E_0} |f_n - f|^s\, d\theta \to 0$ *for* $0 < s < r$.

The first part follows from Theorem 11, the second and third from the main theorem. The remaining last part is a consequence of the second and Theorem 1 (22), $\left[\text{since } \left(\int_{E_0} |f_n - f|^r\, d\theta\right)^{1/r} \leqslant 2K^{1/r}\right]$.

COROLLARY 2. *Suppose that* $r > 1$ *and* $\int_{E_0} |f_n|^r\, d\theta \leqslant G$. *Then* $\int_0^{\theta} f_n d\theta$ *is* u.a.c.

In fact the argument proving $\int_E |f_n|\, d\theta < \epsilon(\mu)$ is independent of the hypothesis $f_n \to f$.

4 . 61. *"Completion"*. Suppose that a function f is defined in a set everywhere dense in E_0, but not everywhere in E_0, and let Q be the exceptional set, θ_0 a point of it. Let us denote by $\Delta(\theta_0, \delta)$ the set of points θ satisfying $|\theta - \theta_0| \leqslant \delta$. We shall define $f(\theta_0)$ to be

$$\lim_{\delta \to 0} [\textit{upper bound of } f(\theta) \textit{ in } \Delta(\theta_0, \delta)],$$

and shall denote by $f_c(\theta)$ the function that results from this completion of the definition. [Taking the lower instead of the upper bound would, of course, produce a different but equally satisfactory completion.]

We often start from a sequence of functions f_n which converges to a limit f except in a null-set in E_0. To ensure that f_c is unique in this case we lay down the conventions that Q is the set in which f_n does not tend to a limit (this phrase being taken strictly), and that f_c is formed from Q and the values of the limit function in $E_0 - Q$.

We observe concerning completed functions :

(i) A point θ_0 belonging to Q is the limit of a sequence $\theta_1, \theta_2, \ldots$ of points belonging to CQ for which

$$f_c(\theta_0) = \lim_{n \to \infty} f(\theta_n) = \lim f_c(\theta_n).$$

(ii) If, by assigning appropriate values to f in Q, it is *possible* to obtain a function continuous in E_0, then f_c *is* this continuous function.

4 . 62. THEOREM 20. *Suppose that $f_n(\theta)$ is* u.a.c. *or* u.b.v. *in E_0 and that $f_n \to f$ in a set everywhere dense in E_0. Then f_c is respectively* a.c. *or* b.v. ; *further, in the* a.c. *case $f_n \to f_c$ everywhere in E_0, and moreover uniformly.*

We consider only the a.c. case; the other is simpler. Consider any e, or (α_n, β_n) $(n = 1, 2, \ldots, N)$. Some of the α_n, β_n may be points of Q. Let us suppose, to fix ideas, that α_1 belongs to Q, but that no other α or β does. Then α_1 is the limit as $\nu \to \infty$ of $\alpha_1^{(\nu)}$, a point of CQ, for which

$$f_c(\alpha_1) = \lim_{\nu \to \infty} f(\alpha_1^{(\nu)}).$$

Then, denoting by e_ν the set of intervals e, modified by the substitution o $\alpha_1^{(\nu)}$ for α_1, we have

$$\sum_1^N |f_c(\beta_n) - f_c(\alpha_n)| = \lim_{\nu \to \infty} \left(|f(\beta_1) - f(\alpha_1^{(\nu)})| + \sum_2^N |f(\beta_n) - f(\alpha_n)| \right)$$

$$= \lim_{\nu \to \infty} \lim_{n \to \infty} t(f_n, e_\nu). \tag{1}$$

Now $me_\nu < 2me = 2\mu$ (say) if $\nu > \nu_0$, and then, by hypothesis, $t(f_n, e_\nu) < \epsilon(\mu)$. Hence

$$\overline{\lim}\, t(f_n, e_\nu) \leqslant \epsilon(\mu) \quad (\nu > \nu_0),$$

and so, from (1),
$$t(f_c, e) \leqslant \epsilon(\mu).$$

This proves the first part.

It remains to prove that $f_n \to f_c$ uniformly in E_0. Given ϵ there exists a δ, independent of θ, such that

(2) $$|f_n(\theta') - f_n(\theta)| < \epsilon \quad \textit{for} \quad |\theta' - \theta| < \delta \textit{ and all } n;$$

moreover (since f_c is continuous), we may suppose also that

(3) $$|f_c(\theta') - f_c(\theta)| < \epsilon \quad \textit{for} \quad |\theta' - \theta| < \delta.$$

Let us divide E_0 into intervals less than δ by ν divisions $\theta_1, \theta_2, \ldots, \theta_\nu$ belonging to CQ. For a given θ let now θ_r be the division point nearest to θ. Then

$$|f_n(\theta) - f_n(\theta_r)| < \epsilon \quad (\text{all } n),$$

$$f_c(\theta) - f_c(\theta_r)| < \epsilon.$$

Finally, by hypothesis,

$$|f_n(\theta_r) - f_c(\theta_r)| < \epsilon \quad (n > n_r).$$

From these three inequalities we have

$$|f_n(\theta) - f_c(\theta)| < 3\epsilon, \quad n > \operatorname{Max}(n_1, n_2, \ldots, n_\nu),$$

and this completes the proof.

4.63. THEOREM 21. *Suppose that $f_n \to f$ in E_0, or in a set everywhere dense in E_0, and that $r > 1$,*

$$f_n = \int_0^\theta g_n \, d\theta + c_n, \qquad \int_{E_0} |g_n|^r \, d\theta \leqslant G.$$

Then $f_c = \int_0^\theta g \, d\theta + c$, *where g is a function such that* $\int_{E_0} |g|^r \, d\theta \leqslant G.$

By Theorem 19, Cor. 2, f_n is u.a.c. Hence, by Theorem 20, f_c is a.c., and so an integral, say the integral of g. Consider now any e and the sum

$$S(f_c) = \sum_1^N \frac{|f_c(\beta_m) - f_c(\alpha_m)|^r}{|\beta_m - \alpha_m|^{r-1}}$$

taken over e. Since $f_n \to f_c$ everywhere (Theorem 20) we have

$$S(f_c) = \lim S(f_n).$$

Hence, since by Theorem 16 (converse part),

$$S(f_n) \leqslant \int_{E_0} |g_n|^r d\theta \leqslant G,$$

have also $S(f_e) \leqslant G$. This being true for all e, Theorem 16 shows that $|g|^r d\theta \leqslant G$.†

4.71. THEOREM 22. *Let $f_n(\theta) = \int_0^\theta g_n d\theta + c_n$ $(n = 1, 2, \ldots)$ be a uniformly bounded and* u.a.c. *sequence of functions. Then there exists a subsequence $f_{n_r}(\theta)$, $r = 1, 2, \ldots,$ which converges uniformly to a function $f(\theta) = \int_0^\theta g d\theta + c$, where g is some function of class L. If further $r > 1$ and $\int_{E_0} |g_n|^r d\theta \leqslant G$, all this happens, and also $\int_{E_0} |g|^r d\theta \leqslant G$.*

Let $\theta_1, \theta_2, \ldots$ be the rational multiples of π in E_0, arranged in any way as a progression. By Theorem 5 there exists a sequence $n_1, n_2, \ldots$ and numbers $a_1, a_2, \ldots$ such that, for each m,

$$\phi_r(\theta_m) = f_{n_r}(\theta_m) \to a_m$$

as $r \to \infty$. The sequence $\phi_r(\theta)$ is u.a.c. (*a fortiori*, since f_n is). It follows from Theorem 20 that a_m is the value at $\theta = \theta_m$ of a certain a.c. function $f = \int_0^\theta g d\theta + c$, and that $f_{n_r} \to f$ uniformly in E_0 as $r \to \infty$. This proves the first part: the last follows from Theorem 21.

4.8. *Convergence in mean.*

4.81. *Convergence on the average.* Let $f_1, f_2, \ldots$ be a sequence of functions, f another function, and let $E(\nu, \epsilon)$ be the set (in E_0) in which $|f_\nu - f| > \epsilon$. If now, for each fixed ϵ, $\lim_{\nu \to \infty} mE(\nu, \epsilon) = 0$, we say that f_n converges on the average to f (in E_0).

THEOREM 23. *Let $f_1, f_2, \ldots$ be a sequence of functions, each finite* p.p., *$E(\mu, \nu, \epsilon)$ the set (in E_0) in which $|f_\mu - f_\nu| > \epsilon$, and suppose that for each fixed ϵ $mE(\mu, \nu, \epsilon) \to 0$ as μ, ν tend independently to ∞. Then there exists an f and a subsequence (f_{n_r}), such that $f_{n_r} \to f$* p.p. *as $r \to \infty$. Also any two such f are equivalent, and f_n converges on the average to f.*

† Theorem 16 has fulfilled its purpose in establishing Theorem 21, and is never used again.

Conversely, if f_n converges on the average to an f finite p.p., *then all the other events happen*.

If $mE(\mu, \nu, \epsilon) \to 0$, then, given any convergent series $\Sigma\epsilon_n$, we can find an (increasing) sequence $n_1, n_2, \ldots$ such that

$$mE(n_r, n_{r+1}, \epsilon_r) < \epsilon_r.$$

Consider now the series

$$\sum_{r=1}^{\infty}(f_{n_r} - f_{n_{r+1}}) = \Sigma u_r.$$

If θ does not belong to $E_\nu = \sum_{r=\nu}^{\infty} E(n_r, n_{r+1}, \epsilon_r)$ we have $|u_r| \leqslant \epsilon_r$ for $r > \nu$, and the series converges (absolutely). Since $mE_\nu < \sum_{\nu}^{\infty}\epsilon_r \to 0$ as $\nu \to \infty$ the set in which the series diverges has arbitrarily small measure, and consequently has measure 0. That is, $f_{n_r} \to f$ p.p. (for some limit function f).

Next, given ϵ, η, we have

(1) $$|f_n - f_{n_r}| \leqslant \epsilon$$

except in a set $E_1 = E(n, n_r, \epsilon)$, where $mE_1 < \eta$ provided $n \geqslant \nu(\epsilon, \eta)$, $r \geqslant r_0(\epsilon, \eta)$. Also, by Theorem 9, $f_{n_r} \to f$ uniformly in CE_2, where $mE_2 < \eta$, so that

(2) $$|f_{n_r} - f| < \epsilon \quad (r \geqslant r_1)$$

except in E_2. Combining (1) and (2) for $r = \operatorname{Max}(r_0, r_1)$ we have $|f_n - f| < 2\epsilon$ except in a set (depending on n) of measure less than 2η. Hence f_n converges on the average to f.

Suppose now that f and f^* are two "limit-functions". Then we have simultaneously

$$|f_n - f| < \epsilon, \quad |f_n - f^*| < \epsilon,$$

except in a set $E_n(\epsilon)$ whose measure tends to 0 as $n \to \infty$. Since $|f - f^*| < 2\epsilon$ except in E_n it follows that the set $E(\epsilon)$ in which $|f - f^*| \geqslant 2\epsilon$ has arbitrarily small measure, and so measure zero. The set, $\lim\limits_{\epsilon \to 0} E(\epsilon)$, in which $f^* \neq f$ therefore also has measure zero, and $f^* \equiv f$.

Finally, since

$$E(\mu, \nu, 2\epsilon) \subset E(\mu, \epsilon) + E(\nu, \epsilon),$$

$mE(\mu, \nu, \epsilon) \to 0$ for each ϵ as $\mu, \nu \to \infty$ if $mE(\nu, \epsilon) \to 0$ for fixed ϵ as $\nu \to \infty$. This proves the last part.

4.82. *Strong and weak convergence.* If $\lambda > 0$, f is of class L^λ and $\int_{E_0} |f_n - f|^\lambda d\theta \to 0$ as $n \to \infty$, we say that f_n converges strongly to f, with index λ.

The simplest case of strong convergence (and, incidentally, of convergence on the average) in which we do not also have $f_n \to f$ (p.p., or indeed anywhere) is afforded by an f_n whose graph is the real axis except for an interval I_n, of centre P_n and length $l_n = o(1)$ (not *too* small), upon which there is a steep peak, the set of P_n being everywhere dense in E_0. For example, we may take the point $2\pi\{n\sqrt{2}\}$ (where $\{x\}$ denotes x less the integer nearest to x) for P_n, n^{-1} for l_n, and $f_n = n^{\frac{1}{2}}$ or 1 respectively in I_n (and zero elsewhere). Evidently $\int_{E_0} f_n^r d\theta \to 0$ if $r < 2$ in the first case and for all r in the second; f_n converges strongly to 0 with index r. On the other hand, familiar facts about rational approximations show that any θ is interior to an infinity of I_n (and, of course, exterior to another infinity), so that $\lim f_n$ exists for no θ.

If $r > 1$,

(1) $$\int_{E_0} |f_n|^r d\theta \leqslant G,$$

and

(2) *there exists an f such that for each θ of E_0*

$$\int_0^\theta f_n d\theta \to \int_0^\theta f d\theta,$$

we say that f_n converges weakly to f, with index r. Two "limit-functions" of the same sequence are necessarily equivalent, since the integral of their difference is identically zero.

We observe in passing that, subject to (1), (2) must hold for all θ if it holds in a set of θ everywhere dense in θ (Theorem 21).

The simplest example of weak convergence is $f_n = \sin n\theta$. f_n converges weakly to 0 with any index $r > 1$ (by Theorem 8).

4.83. THEOREM 24. *If $f_n \to f$ p.p. in E_0, $r > 1$, and $\int_{E_0} |f_n|^r d\theta \leqslant G$, then $\int_{E_0} |f_n - f|\, d\theta \to 0$ and (a fortiori) f_n converges weakly to f.*

This is merely a restatement of Theorem 19, Cor. 1.

THEOREM 25. *If f_n converges strongly to f with index $r > 1$, then f_n converges weakly to f with index r. Also $\int_E |f_n|^r d\theta \to \int_E |f|^r d\theta$ as $n \to \infty$ if $E \subset E_0$.*

The last part is a case of Theorem 2 (18), and it shows incidentally that $\int_{E_0} |f_n|^r d\theta$ is bounded. Also $\int_{E_0} |f_n - f| \, d\theta \to 0$, by Theorem 1 (10).

4.84. THEOREM 26. *If f_n converges weakly to f with index r and constant G, then*

$$\int_{E_0} |f|^r d\theta \leqslant \varliminf_{n \to \infty} \int_{E_0} |f_n|^r d\theta \leqslant G.$$

It follows first from Theorem 21 [with f_n for g_n] that $\int_{E_0} |f|^r d\theta \leqslant G$. Next, if $\varliminf \int_{E_0} |f_n|^r d\theta = l$, there exists a sequence (n_m) giving $\int_{E_0} |f_{n_m}|^r d\theta \to l$, or

$$\int_{E_0} |f_{n_m}|^r d\theta < l + \epsilon \quad (m > m_0).$$

Since the sequence f_{n_m} $(m > m_0)$ converges weakly to f with constant $l + \epsilon$ we have $\int_{E_0} |f|^r d\theta \leqslant l + \epsilon$, whence the result.

It is false that

$$\int_{E_0} |f|^r d\theta = \lim \int_{E_0} |f_n|^r d\theta;$$

a *Gegenbeispiel* is $f_n = \sin n\theta$, $f = 0$.

4.85. THEOREM 27. *If f_n converges weakly to f with index r and g belongs to class $L^{r'}$ then*

$$\int_E f_n g \, d\theta \to \int_E fg \, d\theta \quad (E \subset E_0)$$

as $n \to \infty$.

We may suppose (by taking $g = 0$ in $E_0 - E$) that $E = E_0$. By Theorem 6 there exists a step-function ϕ in E_0 for which

$$\int_{E_0} |g - \phi|^{r'} d\theta < \delta.$$

By Theorem 26 f is of class L^r; hence, by Theorem 1, $f_n g$ and fg are of class L. Then

$$\left| \int_{E_0} f_n g \, d\theta - \int_{E_0} fg \, d\theta \right| \leqslant \left| \int_{E_0} (f_n - f) \phi \, d\theta \right| + \left| \int_{E_0} (f_n - f)(g - \phi) \, d\theta \right|.$$

Since ϕ is constant in stretches the first integral on the right is of the form

$$\Sigma c_r \left\{ \int_0^{\beta_r} (f_n - f) \, d\theta - \int_0^{\alpha_r} (f_n - f) \, d\theta \right\},$$

and tends to 0 in virtue of the hypothesis. The second term on the right does not exceed

$$\left(\int_{E_0} |f-f_n|^r d\theta\right)^{1/r} \left(\int_{E_0} |g-\phi|^{r'} d\theta\right)^{1/r'},$$

of which the first factor does not exceed

$$\left(\int_{E_0} |f|^r d\theta\right)^{1/r} + \left(\int_{E_0} |f_n|^r d\theta\right)^{1/r} \leqslant 2G^{1/r}$$

and the second does not exceed $\delta^{1/r'}$. It follows that

$$\left|\int_{E_0} f_n g\, d\theta - \int_{E_0} fg\, d\theta\right| < o(1) + \epsilon(\delta),$$

and so that

$$\int_{E_0} f_n g\, d\theta \to \int_{E_0} fg\, d\theta.$$

4 . 86. THEOREM 28. *Suppose that* $r > 1$, $\int_{E_0} |f_n|^r d\theta \leqslant G$. *Then there exists a subsequence* f_{n_m} *converging weakly* (r) *to some* f $\left(\textit{for which } \int_{E_0} |f|^r d\theta \leqslant G\right)$, *the convergence of* $\int_0^\theta f_{n_m} d\theta$ *to* $\int_0^\theta f d\theta$ *being moreover uniform.*

$F_n(\theta) = \int_0^\theta f_n d\theta$ is the general term of a u.a.c. sequence (Theorem 19, Cor. 2) ; also

$$|F_n| \leqslant \int_{-\pi}^{\pi} |f_n|\, d\theta \leqslant 2\pi M_r(f_n) \leqslant 2\pi G^{1/r}.$$

By Theorem 22 there exists a subsequence F_{n_m} converging uniformly to some $F(\theta) = \int_0^\theta f d\theta$. [The constant c is zero since $F_n(0) = 0$, and so $F(0) = 0$.] This proves all but the parenthesis, and that follows from Theorem 26.

4 . 87. THEOREM 29. *Let* (f_n) *be a sequence of functions, each of class* $\lambda > 0$, *and suppose that for each given* ϵ

$$\int_{E_0} |f_\mu - f_\nu|^\lambda d\theta < \epsilon \quad \left(\mu, \nu \geqslant \nu_0(\epsilon)\right).$$

Then,

(1) *there exists a subsequence* f_{n_m} *converging* p.p. *to some* f *of class* λ.

(2) *any two limit functions* f, f^* *of this kind are equivalent,*

(3) f_n *converges strongly* (λ) *to* f,

(4) f_n *converges on the average to* f.

With the notation of Theorem 23 we have

$$mE(\mu, \nu, \eta) \leqslant \eta^{-\lambda} \int_{E(\mu, \nu, \eta)} | f_\mu - f_\nu |^\lambda d\theta < \epsilon \eta^{-\lambda} \quad \big(\mu, \nu > \nu_0(\epsilon)\big).$$

Hence for each η $mE(\mu, \nu, \eta) \to 0$ as $\mu, \nu \to \infty$ and (1), (2), and (4) follows from Theorem 23, except that we cannot conclude that f is of class L^λ. Finally, if $n > \nu_0$,

$$\int_{E_0} | f - f_n |^\lambda d\theta = \int \lim_{m \to \infty} | f_{n_m} - f_n |^\lambda d\theta$$

$$\leqslant \underline{\lim} \int | f_{n_m} - f_n |^\lambda d\theta \quad \text{(Theorem 11)}$$

$$< \int \epsilon \, d\theta = 2\pi\epsilon,$$

by hypothesis: f_n converges strongly (λ) to f. Incidentally this shows that f is of class L^λ, and the proof is completed.

If $\lambda = k \geqslant 1$ (the most important case) it is possible to give an alternative proof of (1), (2), (3). This does not depend on the idea of average convergence, or on Theorem 9 (but results from combining the *argument* of § 4.81 and the proof just given).

Let $\Sigma\epsilon_r$ be a convergent series. In virtue of our hypothesis we can find an increasing sequence (n_r) such that

$$\frac{1}{2\pi}\int_{E_0} | u_r |^k d\theta = \frac{1}{2\pi}\int_{E_0} | f_{n_{r+1}} - f_{n_r} |^k d\theta < \epsilon_r^k.$$

Then
$$\frac{1}{2\pi}\int_{E_0} | u_r | \, d\theta \leqslant \epsilon_r$$

and
$$\lim_{N \to \infty} \int_{E_0} \Big(\sum_1^N | u_r |\Big) d\theta = \sum_1^\infty \int_{E_0} | u_r | \, d\theta \leqslant 2\pi \Sigma\epsilon_r.$$

By Theorem 11

$$\int_{E_0} \Big(\sum^\infty | u_r |\Big) d\theta = \int \lim_{N \to \infty} \leqslant \underline{\lim} \int \leqslant 2\pi\Sigma\epsilon_r < \infty,$$

the integrand is finite p.p. and Σu_r is absolutely convergent p.p. In particular $f_n \to f$ p.p. (for some f). But now

$$\int_{E_0} | f - f_n |^k d\theta = \int_{E_0} \lim_{r \to \infty} | f_{n_r} - f_n |^k d\theta \leqslant \underline{\lim} \int < \epsilon \quad (n > n_0$$

and f_n converges strongly (k) to f. Thus (1) and (3) are proved, and (2) is a case of Theorem 2 (19).

Finally, if $\lambda = r > 1$, there is another proof of (3) which is worth our notice.

If $\nu > \nu_0(1)$ we have

$$\int_{E_0} |f_\nu - f_{\nu_0}|^r d\theta \leqslant 1, \quad \int_{E_0} |f_\nu|^r d\theta \leqslant A_r + A_r \int_{E_0} |f_{\nu_0}|^r d\theta,$$

so that we have, for all n,

$$\int_{E_0} |f_n|^r d\theta \leqslant G.$$

By Theorem 28 there exists a subsequence f_{n_m} converging weakly to some f. Since $f_{n_m} - f_n$ converges weakly to $f - f_n$, Theorem 26 gives

$$\int_{E_0} |f - f_n|^r d\theta \leqslant \varliminf_{m \to \infty} \int_{E_0} |f_{n_m} - f_n|^r d\theta < \epsilon \quad (n > n_0),$$

and this proves (3).

5. *Fourier series.*

5.1. *Notation.* In what follows we suppose always, unless the contrary is stated, that $f(\theta)$ is of class L and has the period 2π. The constants

(1) $$\begin{cases} a_n = a_n(f) = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta \\ b_n = b_n(f) = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta \end{cases} \qquad (n \geqslant 0)$$

are called the Fourier cosine and sine coefficients of f. For negative n we define

(2) $$a_n = a_{-n}, \quad b_n = -b_{-n} \quad (b_0 = 0)$$

(and these equations are then valid for $-\infty < n < \infty$). The constants

(3) $$c_n = c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-n\theta i} d\theta \quad (-\infty < n < \infty)$$

we call the complex Fourier coefficients of f (even if a_n, b_n are complex). Clearly

(4) $$c_n = \tfrac{1}{2}(a_n - ib_n).$$

We indicate the relations of the a_n, b_n, c_n to f by

$$(5) \qquad f \sim \tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta).$$

$$(6) \qquad f \sim \sum_{-\infty}^{\infty} c_n e^{n\theta i},$$

in which there is no implication that the "Fourier series" on the right are convergent, or that if they are convergent the sums are equal to $f(\theta)$. If $c_n = 0$ for $n < 0$ the "Fourier series" is said to be a "Fourier power series". The condition for this in terms of the a_n, b_n is that $b_n = ia_n$ for all positive n.

We write also

$$(7) \qquad \alpha_n = \alpha_n(\theta) = a_n \cos n\theta + b_n \sin n\theta \quad (-\infty < n < \infty).$$

We have the following identities, in which we suppose $n > 0$:

$$(8) \qquad s_n = s_n(\theta) = \tfrac{1}{2}a_0 + \sum_1^n \alpha_m(\theta) = \tfrac{1}{2}a_0 + \sum_{m=1}^n (a_m \cos m\theta + b_m \sin m\theta)$$
$$= \sum_{-n}^n c_m e^{m\theta i}.$$

$$(9) \qquad \sigma_n = \sigma_n(\theta) = \frac{s_0(\theta) + s_1(\theta) + \ldots + s_{n-1}(\theta)}{n} = \tfrac{1}{2}a_0 + \sum_{m=1}^n \left(1 - \frac{m}{n}\right) \alpha_m$$
$$= \sum_{-n}^n \left(1 - \frac{|m|}{n}\right) c_m e^{m\theta i}.$$

Considered as derived from an f of which the a's, b's, and c's are Fourier coefficients, s_n and σ_n are also written $s_n(\theta, f)$, $\sigma_n(\theta, f)$.

We observe that if P_N is a "trigonometrical polynomial" $\sum_{-N}^N u_m e^{m\theta i}$ then

$$s_n(P_N) = P_n, \quad \sigma_n(P_N) = \sum_{-n}^n \left(1 - \frac{|m|}{n}\right) u_m e^{m\theta i} \quad (N \geqslant n > 0)$$

In particular we have

$$s_n\{s_N(f)\} = s_n(f), \quad \sigma_n\{s_N(f)\} = \sigma_n(f) \quad \text{if} \quad N \geqslant n > 0.$$

For, *e.g.*, $$s_n(P_N) = \sum_{m=-n}^n \left\{\frac{1}{2\pi}\int_{-\pi}^\pi \left(\sum_{-N}^N u_r e^{r\theta i}\right) e^{-m\theta i} d\theta\right\} e^{m\theta i},$$

and we have only to integrate term by term. For a similar reason a

uniformly convergent trigonometrical series is the Fourier series of its sum.

We write further

(10) $$\phi(t) = \phi(t, \theta, s) = \tfrac{1}{2}\{f(\theta+t)+f(\theta-t)-2s\},$$

(11) $$\phi_0(t) = \tfrac{1}{2}\{f(\theta+t)+f(\theta-t)-2f(\theta)\},$$

(12) $$\Phi(t) = \int_0^t \phi(u)\,du, \quad \Phi_0(t) = \int_0^t \phi_0(u)\,du,$$

(13) $$\Phi^*(t) = \int_0^{|t|} |\phi(u)|\,du, \quad \Phi^*(t) = \int_0^{|t|} |\phi_0(u)|\,du.$$

5.21. THEOREM 30. *If f is of class L then a_n, b_n, c_n tend to zero as $n \to \infty$. If f is* b.v. *they are of the form $O(n^{-1})$; if f is* a.c. *they are of the form $o(n^{-1})$. In any case*

$$|c_n| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|\,d\theta = I, \quad |a_n| \leqslant 2I, \quad |b_n| \leqslant 2I.$$

The first part follows from Theorem 8. For the second we may suppose f real, positive, and decreasing (Theorem 14)†, and then, by the second mean value theorem,

$$\pi a_n = f(-\pi)\int_{-\pi}^{\theta_1} \cos n\theta\,d\theta,$$

where $-\pi \leqslant \theta_1 \leqslant \pi$. This gives $a_n = O(n^{-1})$, and the result for b_n is proved similarly.

If f is a.c. it is an integral: say $\int_0^\theta g(t)\,dt + c$. Then for $n > 0$,

$$\int_{-\pi}^{\pi} f \cos n\theta\,d\theta = \int_{-\pi}^{\pi}\left(\int_0^\theta g(t)\,dt\right)\cos n\theta\,d\theta$$

$$= \left[\frac{\sin n\theta}{n}\int_0^\theta g\,dt\right]_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi} g(\theta)\sin n\theta\,d\theta = 0 + o(n^{-1}),$$

by the first part.

The last part follows at once from the definitions of a_n, b_n, c_n.

5.22. THEOREM 31. *Suppose that (f is of class L and) $f \sim \Sigma c_m e^{m\theta i}$, and that $F(\theta) = \int_0^\theta f(t)\,dt - c_0\theta$. Then*

$$c_m(F) = \frac{c_m(f)}{mi} \quad (m \neq 0).$$

† For once, of course, we drop the convention $f(-\pi) = f(\pi)$.

For

$$c_m(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-m\theta i}\, d\theta = \frac{1}{2\pi}\Big[F(\theta)\, e^{-m\theta i}\Big]_{-\pi}^{\pi} + \frac{mi}{2\pi}\int_{-\pi}^{\pi} F(\theta)\, e^{-m\theta i}\, d$$

$$= 0 + mi\, c_m(F) = mi\, c_m(F),$$

since $$F(\pi) - F(-\pi) = \int^{\pi} f(\theta)\, d\theta - 2\pi c_0 = 0.$$

5.31. THEOREM 32. *Suppose that f belongs to class L. Then for any s*

(1) $$s_n(\theta) - s = \frac{1}{2\pi}\int_{-\pi}^{\pi} \{f(\theta+t) - s\}\, Q(t)\, dt = \frac{1}{\pi}\int_0^{\pi} \phi(t)\, Q(t)\, dt,$$

(2) $$\sigma_n(\theta) - s = \frac{1}{2\pi}\int_{-\pi}^{\pi} \{f(\theta+t) - s\}\, R(t)\, dt = \frac{1}{\pi}\int_0^{\pi} \phi(t)\, R(t)\, dt,$$

where

(3) $$Q(t) = Q_n(t) = \frac{\sin (n+\frac{1}{2})\, t}{\sin \frac{1}{2} t}, \quad R(t) = R_n(t) = \frac{\sin^2 \frac{1}{2} n t}{n \sin^2 \frac{1}{2} t},$$

so that Q and R are even functions of t, $R(t) \geqslant 0$, and

(4) $$\frac{1}{\pi}\int_0^{\pi} Q(t)\, dt = \frac{1}{\pi}\int_0^{\pi} R(t)\, dt = 1.$$

We have, writing $u_m = 1 - |m|/n$ (and recalling that f is periodic),

$$\sigma_n(\theta) = \sum_{-n}^{n} u_m c_m e^{m\theta i} = \sum_{-n}^{n} u_m e^{m\theta i} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-mti}\, dt$$

$$= \frac{1}{2\pi} \sum_{-n}^{n} u_m \int_{-\pi}^{\pi} f(\theta+t)\, e^{-mti}\, dt,$$

and so, since $\sigma_n - s = \sigma_n(f-s)$,

$$\sigma_n(\theta) - s = \frac{1}{2\pi} \sum_{-n}^{n} u_m \int_{-\pi}^{\pi} \{f(\theta+t) - s\}\, e^{-mti}\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \{f(\theta+t) - s\}\, R(t)\, dt,$$

where $R(t) = \sum_{-n}^{n} u_m e^{-mti}$. This sum reduces to the form in (3) by elementary calculations. The case of s_n is similar [we have $u_m = 1$ $(m \neq 0)$, $u_0 = \frac{1}{2}$]. We thus obtain the first forms in (1) and (2). The second forms are derived by associating the values of the integrand for $-t$ and $+t$. Finally, if we take the special f for which $f - s = 1$ we have $\phi = 1$ in (1) and (2), and this leads to (4).

5.32. THEOREM 33. *The upper and lower limits as $n \to \infty$ of s_n and σ_n are unaffected by any change in $f(\theta)$ outside an arbitrarily small interval $(\theta-\delta, \theta+\delta)$, provided only that f remains of class L in E_0.*

In fact

$$(1) \qquad \left| \int_\delta^\pi \{f(\theta+t)+f(\theta-t)\}\, Q(t)\, dt \right| = \left| \int_\delta^\pi \chi(t) \sin(n+\tfrac{1}{2})\, t\, dt \right|,$$

when $\chi(t) = \{f(\theta+t)+f(\theta-t)\} \operatorname{cosec} \frac{1}{2}t$; and since (for $\delta < \pi$)

$$\int_\delta^\pi |\chi|\, dt \leqslant \operatorname{cosec} \tfrac{1}{2}\delta \int_\delta^\pi \big(|f(\theta+t)| + |f(\theta-t)|\big)\, dt$$
$$\leqslant 2 \operatorname{cosec} \tfrac{1}{2}\delta \int_{-\pi}^\pi |f(t)|\, dt < \infty,$$

the right-hand side of (1) is $o(1)$ in virtue of Theorem 8. Also (for σ_n),

$$(2) \quad \left| \int_\delta^\pi \phi(t)\, R(t)\, dt \right| \leqslant \frac{1}{n} \int_\delta^\pi |\chi| \operatorname{cosec} \tfrac{1}{2}t\, dt \leqslant \frac{2 \operatorname{cosec}^2 \frac{1}{2}\delta}{n} \int_{-\delta}^\pi |f(t)|\, dt = o(1)$$

5 . 33. When a trigonometrical series is uniformly convergent it is, as we observed in § 5 . 1, the Fourier series of the function $f(\theta)$ that is its sum, and then $f(\theta)$ is represented approximately by the polynomial $s_n(\theta)$ when n is large. This is a desirable state of things, but in general nothing so simple is true. In the first place the c_n are unaltered if we alter $f(\theta)$ at a single point; hence the series and the *value* of $f(\theta)$ have, at an *assigned* point, no particular connexion. This fact suggests negatively, and Theorem 33 suggests positively, that the "average of f at θ", that is to say

$$\lim_{h\to 0} \frac{1}{2h} \int_{-h}^{h} f(\theta+t)\, dt,$$

or the limit of some higher mean, is the number, if any, to be associated with the series at the point θ. Secondly, the series may fail to converge at some or all points θ; we may then inquire whether the arithmetic mean $\sigma_n(\theta)$, or some higher mean of $s_n(\theta)$, converges to a limit in place of $s_n(\theta)$. What is actually true (though the proof would take us too far) is that if some average of f exists at θ, then some mean of $s_n(\theta)$ converges, and to that value. Now for most important purposes the convergence of a mean of s_n is quite as good as that of s_n itself. As for the distinction between $f(\theta)$ and its average, there are two principal cases. In one f is continuous, and so everywhere identical with its average; in this case the facts are that $\sigma_n(\theta) \to f(\theta)$ uniformly. In the other we are satisfied with convergence p.p., and it is a classical result of Lebesgue theory that any $f(\theta)$ (of class L) has p.p. an average equal to itself. Here the facts are that $\sigma_n(\theta) \to f(\theta)$ p.p.

The "convergence problem" proper for Fourier series, that is, the question under what circumstances $s_n(\theta)$ tends to a limit (for a particular

θ or for a class of θ), forms for the most part a curiously isolated portion of the subject, and we do not need to discuss it.

5.41. THEOREM 34 (Fejér). *If $f(\theta)$ (or any function equivalent to it) is bounded above, or below, or both, then, for every value of n, $\sigma_n(\theta)$ has the same bound or bounds as f. If $f(t) \leqslant C(\delta)$ in $0 < |t-\theta| < \delta$ and $C = \varliminf_{\delta \to 0} C(\delta)$, then $\varlimsup \sigma_n(\theta) \leqslant C$, and similarly for $\varliminf \sigma_n(\theta)$ and a lower bound $C' = \varliminf_{\delta \to 0} C'(\delta)$. If $f(\theta)$ is continuous in E_0, $\sigma_n(\theta)$ is continuous in E_0 uniformly in n.*

Suppose, *e.g.*, that $f(\theta) \leqslant a$. Then, by Theorem 32,

$$\sigma_n(\theta)-a = \frac{1}{2\pi}\int_{-\pi}^{\pi} \{f(\theta+t)-a\}\, R(t)\, dt \leqslant 0,$$

since the integrand is non-positive. In the second part, *e.g.* in the case of an upper bound, we may suppose, by Theorem 33, that $f \leqslant C(\delta)$ for *all* θ. Then $\varlimsup \sigma_n(\theta) \leqslant C(\delta)$, and so $\varlimsup \sigma_n(\theta) \leqslant C$. The last part follows from the first since $\Delta\sigma_n = \sigma_n(\theta+h)-\sigma_n(\theta) = \sigma_n\{\Delta f\}$.

5.42. THEOREM 35. *At a point of continuity of $f(\theta)$, $\sigma_n(\theta) \to f(\theta)$ as $n \to \infty$. If f is continuous in E_0†, then $\sigma_n(\theta) \to f(\theta)$ uniformly in E_0.*

We have $$\pi(\sigma_n - f) = \int_0^{\delta} \phi_0(t)\, R(t)\, dt + \int_{\delta}^{\pi} \phi_0(t)\, R(t)\, dt.$$

Since in $\delta \leqslant t \leqslant \pi$

$$|\phi_0(t)|\, R(t) \leqslant \{|f(\theta+t)| + |f(\theta-t)| + 2|f(\theta)|\}\, \frac{A(\delta)}{n},$$

we have $$\left|\int_{\delta}^{\pi}\right| \leqslant \frac{A(\delta)}{n}\left\{A\,|f(\theta)| + A\int_{-\pi}^{\pi} |f(t)|\, dt\right\} = o(1),$$

and uniformly in the second case of the theorem. We have further, in the first case, $|\phi| < \epsilon$ in $0 \leqslant t \leqslant \delta$ by choice of δ, and in the second case this result holds also uniformly in θ. Hence

$$|\sigma_n - f| < \epsilon + o(1) \quad (n > n_0)$$

in either case, and uniformly in θ in the second case.

5.43. We denote by Λ the set of θ in which, for *every* c,

$$\frac{d}{dt} \quad |f(\theta+t)-c|\, dt\} = |f(\theta)-c$$

† Remember we are supposing (as usual) that $f(-\pi) = f(\pi)$.

at $t=0$. It is a classical theorem of the Lebesgue theory that $C\Lambda$ is null. For any point θ of Λ we have, as $t \to 0$ (by positive and negative values),

$$(1) \qquad \int_0^t |f(\theta+t)-f(\theta)|\,dt = o(t), \quad \Phi_0^*(t) = o(t), \quad \Phi_0(t) = o(t).$$

THEOREM 36. *For any values of θ and s for which $\Phi^*(t) = o(t)$ we have $\sigma_n(\theta) \to s$. In particular, at every point of Λ, and therefore* p.p., *we have $\sigma_n(\theta) \to f(\theta)$.*

By Theorem 33 it is enough to prove that

$$\left|\int_0^\delta \phi(t)\,R(t)\,dt\right| \leqslant \int_0^\delta |\phi(t)|\,R(t)\,dt < \epsilon(\delta) \quad (n > n_0),$$

or that

$$n\int_0^\delta |\phi|\,X(nt)\,dt < \epsilon(\delta), \quad \text{where} \quad X(u) = \left(\frac{\sin u}{u}\right)^2.$$

Now

$$n\int_0^{n^{-1}} |\phi|\,X(nt)\,dt \leqslant n\int_0^{n^{-1}} |\phi|\,dt = n\Phi^*(n^{-1}) = o(1),$$

by hypothesis, and

$$n\int_{n^{-1}}^\delta |\phi|\,X(nt)\,dt \leqslant n\int_{n^{-1}}^\pi \frac{|\phi|\,dt}{n^2t^2} = \omega(n^{-1})$$

where

$$\frac{\omega(u)}{u} = \int_u^\pi \frac{|\phi|\,dt}{t^2} = \left[\frac{\Phi^*(u)}{u^2}\right]_u^\pi + 2\int_u^\pi \frac{\Phi^*(u)\,du}{u^3} < \frac{\Phi^*(\pi)}{\pi^2} + \int_u^\pi o\left(\frac{1}{u^2}\right)du\dagger$$

$$< O(1) + o(u^{-1})$$

as $u \to 0$. It follows that $\omega(u) = o(1)$, and this proves the theorem.

[Actually $\sigma_n \to s$ if $\Phi(t) = o(t)$ and $\Phi^*(t) = O(t)$, and subject to the second condition the first is necessary as well as sufficient.]

Theorem 36 is a suitable text for the following remarks :

1. Suppose we are concerned with a theorem which asserts that some relation, say the convergence of a sequence, holds p.p., *i.e.* holds except in a null-set Z (and suppose that the premisses of the theorem do not themselves involve exceptional null-sets—if they did, the presence of one in the conclusion would call for no remark). If now we happen to be told what set Z is, what remains to be proved is a pure approximation problem, one of proving that a certain function of n and a fixed parameter

† The contribution of the lower limit of the bracket being negative.

θ tends to a limit under certain conditions (those which express that θ belongs to CZ). The search for an exceptional Z is therefore often the best line of attack on the main theorem.

2. An exceptional set Z is generally† the set of points θ at which *some* function (not always the first to hand) is not the differential coefficient of its integral. This is why so many proofs begin with an integration by parts, and it is why we sometimes [as in Theorem 44, proof (iii)] introduce an integration where it has *prima facie* nothing to do with the problem.

5.51. We proceed now to problems of strong and weak convergence.

THEOREM 37. *Suppose that Φ is periodic, and that*

$$\phi(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(\theta+t)\, H(t)\, dt,$$

where $H(t) \geqslant 0$ and $M_1(H) \leqslant 1$. Then

$$M_k(\phi) \leqslant M_k(\Phi) \quad (k \geqslant 1).$$

We may suppose $M_k(\Phi) < \infty$. Then, by Theorem 1 (4),

$$|\phi(\theta)|^k \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |\Phi(\theta+t)|^k H(t)\, dt \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} H(t)\, dt\right)^{k-1},$$

and the last factor does not exceed unity. Integrating with respect to θ we have

$$M_k^k(\phi) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} M_k^k(\Phi)\, H(t)\, dt \leqslant M_k^k(\Phi).$$

THEOREM 38. *If (f is of class L and) $k \geqslant 1$ we have*

$$\int_{E_0} |\sigma_n(\theta)|^k d\theta \leqslant I = \int_{E_0} |f(\theta)|^k d\theta.$$

This is a case of Theorem 37, with f for Φ, σ_n for ϕ, and R for H.

5.52. THEOREM 39. *Let $k \geqslant 1$ and let f be of class L^k. Then, as $n \to \infty$, σ_n converges strongly to f with index k:*

$$\int_{E_0} |\sigma_n(\theta) - f(\theta)|^k d\theta \to 0.$$

Since $\sigma_n \to f$ p.p., this follows at once from Theorem 38 and Theorem 13 (3).

† Theorem 23 provides an exception.

An alternative proof proceeds as follows (and does not use Theorem 13). We have, by Theorem 37,

$$\int_{-\pi}^{\pi} |\sigma_n(\theta)-f(\theta)|^k \, d\theta = \int_{-\pi}^{\pi} d\theta \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \{f(\theta+t)-f(\theta)\} K(t)\, dt\right)^k$$

$$\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)\, K(t)\, dt = \sigma_n(0, F),$$

where $F(t) = \int_{-\pi}^{\pi} |f(\theta+t)-f(\theta)|^k d\theta$. Since F is continuous at $t=0$ (Theorem 7), $\sigma_n(0, F) = o(1)$ by Theorem 35.

COROLLARY. *If f_1, f_2 have the same Fourier constants, then $f_1 \equiv f_2$.*

For if $f = f_1 - f_2$, $\sigma_n(\theta, f)$ is identically zero, and

$$\int_{-\pi}^{\pi} |\sigma_n - f|\, d\theta \to 0$$

requires $f \equiv 0$.

We use this result in the sequel only in the last of the four proofs of Theorem 43 (it is itself an immediate corollary of that theorem), and the reader who is not interested may ignore it. For proof (iv) of Theorem 43 to be genuinely distinct from the others, however, it is necessary to prove the present result without using "summability" considerations. We give accordingly another of the many existing proofs.

Let $|\alpha| \leqslant \pi$, and let $g(\theta)$ be the function that is unity in $(0, \alpha)$ and zero elsewhere. It is elementary (and well known) that

$$g(\theta) = \sum_0^{\infty} (\gamma_m \cos m\theta + \delta_m \sin m\theta)$$

for all θ of E_0, the sum $\sum_0^{n}$ being uniformly bounded†. Hence

$$\int_0^{\alpha} f(\theta)\, d\theta = \int_{-\pi}^{\pi} fg\, d\theta$$

$$= \lim_{n\to\infty} \int_{-\pi}^{\pi} f \sum_0^{n} (\gamma_m \cos m\theta + \delta_m \sin m\theta)\, d\theta,$$

by Theorem 10,

$$= \lim \sum^{n} (\gamma_m a_m + \delta_m b_m) = \lim 0 = 0.$$

The integral of f is identically zero, and $f \equiv 0$.

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† Actually $\gamma_m = \dfrac{\sin m\alpha}{m\pi}$, $\delta_m = \dfrac{1-\cos m\alpha}{m\pi}$ $(m > 0)$.

5.61. THEOREM 40. *If f_n converges weakly (r) to f, then for each m*

$$\lim_{n\to\infty} c_m(f_n) = c_m(f).$$

We have only to take $g = e^{-m\theta i}$ in Theorem 27.

THEOREM 41. *Suppose that $r > 1$, that $\ldots, c_{-1}, c_0, c_1, \ldots$ are arbitrary numbers, and that $s_n(\theta) = \sum_{-n}^{n} c_m e^{m\theta i}$, $\sigma_n(\theta) = \{s_0(\theta)+\ldots+s_{n-1}(\theta)\}/n$. Suppose further that either*

$$(1) \qquad \frac{1}{2\pi}\int_{E_0} |s_n(\theta)|^r \, d\theta \leqslant G \quad (\text{all } n),$$

or

$$(2) \qquad \frac{1}{2\pi}\int_{E_0} |\sigma_n(\theta)|^r \, d\theta \leqslant G \quad (\text{all } n).$$

Then there exists an $f(\theta)$ of class L^r for which

$$f \sim \sum_{-\infty}^{\infty} c_m e^{m\theta i}, \quad \text{and} \quad \frac{1}{2\pi}\int_{E_0} |f|^r \, d\theta \leqslant G.$$

Since by Theorem 2 (2)

$$\left(\int_{E_0} |\sigma_n|^r \, d\theta\right)^{1/r} \leqslant \frac{1}{n} \sum_{m=0}^{n-1} \left(\int_{E_0} |s_m|^r \, d\theta\right)^{1/r}$$

(2) is a consequence of (1), and it is enough to prove the result for condition (2). By Theorem 28 there exists a subsequence σ_{n_m} converging weakly to some f for which $\frac{1}{2\pi}\int_{E_0} |f|^r \, d\theta \leqslant G$. By Theorem 40

$$c_\nu(f) = \lim_{m\to\infty} c_\nu(\sigma_{n_m}) = \lim\left(1 - \frac{|\nu|}{n_m}\right) c_\nu = c_\nu;$$

that is, $f \sim \Sigma c_\nu e^{\nu\theta i}$.

5.62. THEOREM 42. *Let f be of class L and $r > 1$. Then* (i)

$$(1) \qquad \int_{E_0} |\sigma_n(\theta)|^r \, d\theta \to \int_{E_0} |f(\theta)|^r \, d\theta,$$

whether the right-hand side is finite or infinite; (ii) *the necessary and sufficient condition that $\int_{E_0} |f|^r \, d\theta$ is finite is that $\int_{E_0} |\sigma_n(\theta)|^r \, d\theta$ is bounded.*

If the right-hand side of (1) is finite, then σ_n converges strongly to f by Theorem 39, and (1) follows by Theorem 2 (18) (and *a fortiori*

$\int_{E_0} |\sigma_n|^r d\theta$ is bounded). If $\int_{E_0} |\sigma_n|^r d\theta$ is bounded then $\int_{E_0} |f|^r d\theta$ is finite, by Theorem 41. These facts establish both parts of the theorem.

5.71. THEOREM 43 ("Parseval"). *Suppose that f is of class L. Then*

$$S^2 = \sum_{-\infty}^{\infty} |c_n|^2 = \tfrac{1}{2}\left\{\tfrac{1}{2}|a_0|^2 + \sum_1^{\infty} (|a_n|^2 + |b_n|^2)\right\} = J^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta,$$

where either side may be $+\infty$ (and f may have complex values).

In the first place the two infinite series are equal, since

$$|u+iv|^2 + |u-iv|^2 = 2(|u|^2 + |v|^2)$$

even when u and v are complex, and so

$$|c_n|^2 + |c_{-n}|^2 = \tfrac{1}{2}(|a_n|^2 + |b_n|^2).$$

We need, therefore, only consider the c_n series. We observe further that the theorem is elementary when f is a trigonometrical polynomial $\sum_{-N}^{N} c_n e^{n\theta i}$. For then

$$J^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{-N}^{N} c_n e^{n\theta i} \sum_{-N}^{N} \bar{c}_n e^{-n\theta i}\right) d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{m, n=-N}^{N} c_m \bar{c}_n e^{(m-n)\theta i} d\theta,$$

and the integral of the general term in the last sum is 0 or $2\pi|c_n|^2$ according as $m \neq n$ or $m = n$.

We give four proofs of the theorem, in which the depth of the "summability" theorems appealed to progressively decreases.

Proof (i). By Theorem 42,

$$J^2 = \lim_{n\to\infty} \frac{1}{2\pi} \int |\sigma_n|^2 d\theta$$
$$= \lim \sum_{-n}^{n} \left(1 - \frac{|m|}{n}\right)^2 |c_m|^2$$

(by the special case of a polynomial f). On the one hand the right-hand side is not greater than

$$\lim_{n\to\infty} \sum_{-n}^{n} |c_m|^2 = S^2,$$

and on the other, if N is any fixed integer, it is not less than

$$\lim_{n\to\infty} \sum_{-N}^{N} \left(1 - \frac{|m|}{n}\right)^2 |c_m|^2 = \sum_{-N}^{N} |c_m|^2,$$

so that it is not less than S^2 Hence $J^2 = S^2$.

We need not, however, appeal to so deep a result as Theorem 42. In the first place the inequality $S^2 \leqslant J^2$ is elementary. We have, in fact,

$$0 \leqslant \int_{-\pi}^{\pi} |f-s_n|^2 d\theta = \int_{-\pi}^{\pi} (f-s_n)(\bar{f}-\bar{s}_n)\, d\theta$$

$$= \int_{-\pi}^{\pi} |f|^2 d\theta + \int_{-\pi}^{\pi} |s_n|^2 d\theta - 2 \sum_{-n}^{n} \mathfrak{R} \int_{-\pi}^{\pi} f\bar{c}_m e^{-m\theta i} d\theta$$

$$= 2\pi J^2 + 2\pi \sum_{-n}^{n} |c_m|^2 - 4\pi \sum_{-n}^{n} |c_m|^2.$$

$$\sum_{-n}^{n} |c_m|^2 \leqslant J^2, \quad S^2 \leqslant J^2.$$

It remains to prove the complementary inequality $J^2 \leqslant S^2$, and here our remaining proofs differ.

Proof (ii). Since $|\sigma_n|^2 \to |f|^2$ p.p., we have, by Theorem 11,

$$\underline{\lim}\, \frac{1}{2\pi} \int_{-\pi}^{\pi} |\sigma_n|^2 d\theta \geqslant \frac{1}{2\pi} \int \lim = J^2,$$

which is

$$\underline{\lim} \sum_{-n}^{n} \left(1 - \frac{|m|}{n}\right)^2 |c_m|^2 \geqslant J^2.$$

From this we have, *a fortiori*,

$$J^2 \leqslant \lim \sum_{-n}^{n} |c_m|^2 = S^2.$$

The remaining proofs apply (at least in their most natural form) only when $J < \infty$. In this case $S \leqslant J$ is also finite.

Proof (iii). Here we assume less than the proposition "$\sigma_n \to f$ p.p.", replacing it by Fejér's theorem. The function

$$\phi(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta+t)\overline{f(t)}\, dt \tag{1}$$

is continuous, by Theorem 7, Cor. Also

$$c_m(\phi) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\theta+t)\overline{f(t)}\, e^{-m\theta i}\, dt\, d\theta$$

$$= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\psi)\, e^{-m\psi i}\, \overline{f(t)}\, e^{mti}\, dt\, d\psi$$

$$= c_m(f)\, c_m(\bar{f}) = |c_m|^2.$$

By Theorem 35, therefore,

$$\sum_{-n}^{n} \lim_{n \to \infty} \left(1 - \frac{|m|}{n}\right) |c_m|^2 = \lim \sigma_n(0, \phi) = \phi(0).$$

The left-hand side is S^2 [by an argument used in proof (i)], and $\phi(0) = J^2$.

Proof (iv). We argue, as before, that $c_m(\phi) = |c_m|^2$. Then ϕ and the sum of the absolutely convergent series

$$\sum_{-\infty}^{\infty} |c_m|^2 e^{m\theta i}$$

have the same Fourier series. By the uniqueness theorem (Theorem 39, Cor.),

$$\sum_{-\infty}^{\infty} |c_m|^2 e^{m\theta i} \equiv \phi(\theta).$$

Since both sides are continuous the equivalence involves identity: taking $\theta = 0$ in this we have

$$S^2 = \phi(0) = J^2.$$

COROLLARY. *Suppose that f and g are of class L^2, and that*

$$f \sim \Sigma c_n e^{n\theta i}, \quad g \sim \Sigma c'_n e^{n\theta i}.$$

Then

$$\sum_{-\infty}^{\infty} c_n c'_{-n}, \quad \tfrac{1}{2}\left\{\tfrac{1}{2}a_0 a'_0 + \sum_{1}^{\infty} (a_n a'_n + b_n b'_n)\right\}$$

are absolutely convergent, and have the common sum

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} fg\,d\theta.$$

Since $c_n c'_{-n} + c_{-n} c'_n = \frac{1}{2}(a_n a'_n + b_n b'_n)$ we may confine ourselves to the second series. The absolute convergence follows from that of $\Sigma |c_n|^2$ and $\Sigma |c'_n|^2$. In proving the identity we may suppose (by taking real and imaginary parts and considering separately the four terms thus arising from fg) that f and g are real. The integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (f \pm g)^2 d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f \pm g|^2 d\theta$$

is equal, on the one hand, to

(1) $$\tfrac{1}{2}(a_0 + a'_0)^2 + \sum_{1}^{\infty} \{(a_n \pm a'_n)^2 + (b_n \pm b'_n)^2\},$$

and on the other to

(2) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2 d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} g^2 d\theta \pm \frac{1}{\pi}\int_{-\pi}^{\pi} fg\,d\theta.$$

Hence (1) and (2) are equal, and by subtracting the equation corresponding to the minus sign from that corresponding to the plus sign we obtain the desired result.

5.72. THEOREM 44 (Riesz-Fischer). *Suppose that the c_n are any numbers such that $\sum_{-\infty}^{\infty} |c_n|^2$ converges. Then there exists an f of class L^2 such that $f \sim \sum_{-\infty}^{\infty} c_n e^{n\theta i}$.*

This is an immediate corollary of Theorem 41, with $r = 2$, since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |s_n|^2\, d\theta = \sum_{-n}^{n} |c_m|^2 \leqslant \sum_{-\infty}^{\infty}.$$

On account of the great importance of the theorem, however, we give two more proofs. Let

$$f_n = \sum_{-n}^{n} c_m e^{m\theta i}.$$

Proof (ii). If $n \geqslant m > 0$

$$\int_{-\pi}^{\pi} |f_n - f_m|^2 d\theta = 2\pi \sum_{r=m+1}^{n} (|c_r|^2 + |c_{-r}|^2) \leqslant 2\pi \sum_{m+1}^{\infty} < \epsilon \quad (n \geqslant m > \nu).$$

By Theorem 29, f_n tends strongly, and therefore (Theorem 25) weakly, to an f of class L^2. Hence (Theorem 40)

$$c_r(f) = \lim_{n\to\infty} c_r(f_n) = c_r;$$

that is, $f \sim \Sigma c_m e^{m\theta i}$.

Proof (iii). Let

$$F_n(\theta) = \int_0^{\theta} f_n\, d\theta - c_0\theta = \sum_{-n}^{n}{}' \frac{c_m}{mi}(e^{m\theta i} - 1),$$

$$F(\theta) = \lim_{n\to\infty} F_n(\theta) = \sum_{-\infty}^{\infty}{}' \frac{c_m}{mi}(e^{m\theta i} - 1),$$

the last series being uniformly and absolutely convergent in virtue of

$$\left|\frac{c_m}{mi}(e^{m\theta i} - 1)\right| \leqslant \frac{2\,|c_m|}{|m|} \leqslant \left(|c_m|^2 + \frac{1}{m^2}\right)$$

and the hypothesis.

Since

$$\int_{E_0} |f_n|^2 d\theta = 2\pi \sum_{-n}^{n} |c_m|^2 < K,$$

F_n is u.a.c. (Theorem 19, Cor. 2), and so (Theorem 20) F is of the form $\int_0^{\theta} f^*\, d\theta$. Let now $f = f^* + c_0$; then, since $F(\pi) = F(-\pi)$, we have $c_0(f) = c_0$.

If now $r \neq 0$, we have

$$\begin{aligned}\frac{c_r}{ri} &= c_r(F)\\ &= \frac{c_r(f-c_0)}{ri} = \frac{c_r(f)}{ri} \quad \text{(Theorem 31).}\end{aligned}$$

Thus $c_r(f) = c_r$ for *all* r; or $f \sim \Sigma c_m e^{m\theta i}$.

Finally

$$\int_{-\pi}^{\pi} |f|^2 d\theta = 2\pi \sum_{-\infty}^{\infty} |c_n|^2 < \infty$$

by Theorem 43, so that f is of class L^2. [We may avoid this appeal to Parseval's theorem by using Theorem 21 in place of Theorem 19, Cor. 2, when we may conclude that f^* belongs to class L^2.]

The reader should trace all the proofs back to first principles.

The first proof uses the selection principle, the second the (rather deep) existence of a convergent sub-sequence in strong convergence. The third is the most direct, and constructs (more or less) a definite function f. The underlying idea is that a function of class L^2 and its Fourier series are at least sufficiently closely related for the indefinite integral of f to be equal to the sum of the integrated series (the latter being uniformly convergent); and to construct the integral is to construct the integrand.

The reader may be referred also to the remarks at the end of § 5 . 43.

5 . 81. THEOREM 45. *Suppose that* $\lambda > 0$ *and* f *is of class* L^λ. *Then there exists a trigonometrical polynomial*

$$\phi(\theta) = \sum_{0}^{N} (c_n \cos n\theta + d_n \sin n\theta)$$

such that

(1) $\qquad |f-\phi| < \epsilon$ *except in a set of measure less than* δ,

(2) $$\frac{1}{2\pi}\int_{E_0} |f-\phi|^\lambda d\theta < \epsilon^\lambda.$$

If further f *is bounded above, or below, or both, then the polynomial* ϕ *can be found also with the same bound or bounds.*

This is an important extension of Theorem 6. By that theorem there exists a continuous ψ, with the same bounds (if any) as f, and satisfying

(3) $\qquad |f-\psi| < \epsilon$ *except in a set of measure less than* δ,

(4) $$\frac{1}{2\pi}\int_{E_0} |f-\psi|^\lambda d\theta < \epsilon^\lambda.$$

If now $\phi = \sigma_n(\psi)$ and $n > n_0(\epsilon)$ it follows, by Theorem 34, that ϕ has the same bounds as ψ and therefore as f, and, by Theorem 35, that (if n_0 is suitably chosen)

(5) $$|\phi - \psi| < \epsilon \quad (\text{all } \theta),$$

so that further

(6) $$\frac{1}{2\pi}\int_{E_0} |\phi - \psi|^\lambda d\theta \leqslant \epsilon^\lambda.$$

From (3) and (5) we have

(7) $|f - \phi| < 2\epsilon$ *except in a set of measure less than* δ,

and from (4), (6), and Theorem 2 (16),

(8) $$\frac{1}{2\pi}\int_{E_0} |f - \phi|^\lambda d\theta < A_\lambda \epsilon^\lambda.$$

(7) and (8) complete the proof of the theorem.

COROLLARY. *If f is of class L^λ, then there exists a polynomial*

$$P_N(\theta) = \sum_0^N u_n \theta^n$$

such that $|f - P_N| < \epsilon$ *except in a set of measure less than* δ,

and $$\frac{1}{2\pi}\int_{E_0} |f - P_N|^\lambda d\theta < \epsilon^\lambda.$$

Let $P_N(\theta)$ be the sum of the first $N+1$ terms of the expansion of $\phi(\theta)$ in powers of θ. Then, for fixed ϕ, $|\phi - P_N| < \epsilon$ for all θ provided $N > N_0$. The argument can now be completed as in the proof of the main theorem.

5.91. LEMMA. *Suppose that $f(\theta)$ is of class L, and that $h(\theta)$ is bounded. Then*

$$\lim_{n\to\infty} \int_{E_0} f\{h - \sigma_n(h)\} d\theta = 0.$$

By Theorem 36 $\sigma_n(h) - h \to 0$ p.p.; hence, by Theorem 9, $\sigma_n - h \to 0$ uniformly in CH, where $mH < \delta$. Since further (Theorem 34) $|\sigma_n| \leqslant K$, where K is a bound of $|h|$, so that $|\sigma_n - h| \leqslant 2K$, we have

$$\left|\int_{E_0} f(h - \sigma_n)\, d\theta\right| \leqslant \int_{CH} |f|\,|h - \sigma_n|\, d\theta + 2K\int_H |f|\, d\theta,$$

$$< \int_{CH} |f|\,\epsilon\, d\theta + 2K\epsilon(\delta) \quad (n > n_0)$$

$$\leqslant \epsilon \int_{E_0} |f|\, d\theta + 2K\epsilon(\delta).$$

The right-hand side can be made arbitrarily small, and the result follows.

5 . 92. We can now prove the following extension of Theorem 3 :

THEOREM 46. *Suppose that* $r > 1$, [*f is of class L*†] *and*

$$\left|\frac{1}{2\pi}\int_{E_0} fg\,d\theta\right| \leqslant UV \tag{1}$$

for every trigonometrical polynomial g satisfying

$$\frac{1}{2\pi}\int |g|^{r'}\,d\theta = V^{r'} > 0. \tag{2}$$

Then

$$\frac{1}{2\pi}\int_{E_0} |f|^{r}\,d\theta \leqslant U^{r}. \tag{3}$$

Let $h(\theta)$ be any bounded function satisfying

$$\frac{1}{2\pi}\int_{E_0} |h|^{r'}\,d\theta = V^{r'}. \tag{4}$$

Then I say that

$$\left|\frac{1}{2\pi}\int_{E_0} fh\,d\theta\right| \leqslant UV. \tag{5}$$

By Theorem 38 we have

$$\frac{1}{2\pi}\int_{E_0} |\sigma_n(h)|^{r'}\,d\theta = V_1^{r'},$$

where $V_1 = V_1(n) \leqslant V$. Also $h \not\equiv 0$, so that $V_1 > 0$ $(n > n_0)$. Then $g = \sigma_n(h)V/V_1$ is a trigonometrical polynomial satisfying (2). Hence, by hypothesis,

$$\left|\frac{1}{2\pi}\int_{E_0} f\sigma_n(h)\,d\theta\right| \leqslant \left|\frac{1}{2\pi}\int_{E_0} fg\,d\theta\right| \leqslant UV \quad (n > n_0). \tag{6}$$

But by the lemma, as $n \to \infty$,

$$\frac{1}{2\pi}\int_{E_0} f\{h - \sigma_n(h)\}\,d\theta \to 0, \tag{7}$$

and (5) follows from (6) and (7).

Since (5) holds for every bounded h satisfying (4) the desired result now follows by the original Theorem 3.

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† This is, strictly speaking, implied in (1), since that asserts, among other things, the existence of the integral $\int_{E_0} fg\,d\theta$ for a certain *constant* value of g.

6. *Some theorems of analysis situs.*

6.1. A domain is defined as an open connected set of points. The symbols D, Δ, occasionally d, δ, are used for domains. If z_1, z_2 are any two points of D, it is known that z_1 and z_2 can be joined by a (rectilinear) polygon lying in D. We denote the frontier, or boundary, of D by $F(D)$, or sometimes F. D together with its frontier is denoted by D': this is in accordance with the ordinary notation for derived sets. We use the notation D_- to denote a domain with the property $(D_-)' \subset D$; the "distance" of D_- from $F(D)$ is then positive.

By "closed contour" we mean a closed simple† Jordan curve; by "curve" we mean a Jordan curve. By a curve "extending to ∞" we mean a locus $x = \phi(t)$, $y = \psi(t)$, where $\phi(t)$, $\psi(t)$ are continuous for every t satisfying (say) $0 \leqslant t < 1$, and $\phi^2+\psi^2$ tends to infinity as $t \to 1$.

We take as known concerning a closed contour C that (1) C divides the plane into two domains, an interior and an exterior, and the latter contains all distant points; (2) these domains are simply-connected, a simply-connected *bounded*‡ D being defined by the property that any closed contour consisting of points of D contains in its interior no point of $F(D)$; (3) if $z = 0$ belongs to the interior of C, then

$$\Delta_C \arg z = \text{increment of } \arg z \text{ round } C = 2\pi.$$

We shall take for granted some other, but minor, points of *analysis situs*; in particular, that if a domain Δ is in one-one continuous correspondence (point by point) with a simply-connected domain D, then D also is simply-connected (more generally Δ and D have the same connectivity).

6.2. *Further properties of a closed contour C and its interior D.*

(1) *Let A, B be distinct points of C. Then C divides into C_1, C_2, with no common points except A and B.*

(2) *A cross-cut q (simple curve lying in D except for its end-points) from A to B divides D into q together with simply-connected domains D_1, D_2 without common points. D_1 has $q+C_1$ for frontier, D_2 $q+C_2$.*

These results are well known.

(3) *Given A and B, there is a cross-cut q joining them.*

† *I.e.* without double points.

‡ The case of an unbounded domain being dealt with by the usual processes of interpreting statements about $z = \infty$.

This is equivalent to

(4) *There exists a simple curve L joining an interior point O and A and lying (except for A) in D.*

[For, if L_1, L_2 are simple curves joining O to A and B respectively, let P be the last (on L_2) intersection of L_1 with L_2; we go from A to P on L_1, then from P to B on L_2.]

Proof of (4).—We may suppose that the range of t is $-\pi \leqslant t \leqslant \pi$, that B is $t = \pm\pi$, and that A is $t = 0$. Let P be the variable point t of the curve. The distance $AP = \rho(t)$ is a continuous function of t with $t = 0$ for its only zero. The circle K, with centre A and radius r less than AO and AB, cuts C where $\rho(t) = r$, and this has solutions. (P is inside K when $t = 0$, outside it when $t = \pm\pi$.)

Let $L(r)$, $l(r)$ be the upper and lower bounds of the moduli of the t's for which $\rho(t) = r$. Then, since $\rho(t)$ is continuous, $\rho\big(L(r)\big) = \rho\big(l(r)\big) = r$. Also $0 < l(r) \leqslant L(r)$. and $L(r) \rightarrow 0$ with r. [If $\overline{\lim}\, L = 2\delta > 0$, then $L > \delta$ for some arbitrarily small r; $r = \rho(L) \geqslant \min\limits_{|t| \geqslant \delta} \rho(t) > 0$; and this is false.]

Let $\mu(r) = \operatorname*{Min}\limits_{|t| \geqslant l(r)} \rho(t)$. Clearly $0 < \mu(r) \leqslant r$.

The circumference K contains points of D, for its interior contains the neighbourhood of A and its exterior contains the neighbourhood of B. These points of K interior to D fall into a set of non-overlapping arcs. These, taken closed, are cross-cuts of D. I say that *some of these cross-cuts q'_1, q'_2, ... separate A and B, i.e. join a P_+ (P of positive t) with a P_-.* Suppose not. Then every q' joins a P_+ to a P_+ or a P_- to a P_-. Let t_1, t'_1 be the ends of q'_1. Let C_1 be the curve derived from C by replacing the piece from t_1 to t'_1 by q'_1, described by a parameter running from t_1 to t'_1. Let q'_{n_2} be the first q' that is a cross-cut of C_1 (or D_1), and derive C_2 similarly, and so on.† The infinite process yields a Jordan curve C_ω, since for any t between $\pm\pi$ there exists a definite point, continuous in t. Also A belongs to C_ω, since A never lies on a rejected piece (the ends of rejected pieces have t's of the same sign and magnitude $\geqslant l$). So, clearly, does B. C_ω is clearly simple, and encloses a D_ω. Also $C_\omega \subset D'$; hence $D_\omega \subset D$. But now the circumference makes no cross-cuts in D_ω, and this is impossible.

Let q_1, q_2, ... be the set of cross-cuts separating A and B; $q = q(r)$

† No q' joins a t between t_1 and t'_1 to one outside this range, by (2) and the fact that two q' cannot intersect.

that one which C, described positively from A, first meets; and let

$$d = d(A, r) = d(r)$$

be the domain, with A on the boundary, that q cuts off [see (2)]; $m(r)$ the upper bound of the distance of a point of $d(r)$ from A. Then I say

$$m(r) \to 0 \quad as \quad r \to 0. \tag{5}$$

For if C_1+q is the boundary of $d(r)$, *C_1 corresponds to a range of t of type* $-\tau \leqslant t \leqslant \tau'$; also the extreme t correspond to the junctions with q, for which $\rho(t) = r$. Therefore $|t| \leqslant L(r)$ on C_1, and

$$\operatorname*{Max}_{C_1} \rho(t) \leqslant \operatorname*{Max}_{|t| \leqslant L(r)} \rho(t) \to 0,$$

$$\operatorname*{Max}_{d} AP = \operatorname*{Max}_{C_1+q} AP \to 0, \quad m(r) \to 0.$$

Observing now that

$$\{\textit{distance of } A \textit{ from a point of } D - d(r)\} \geqslant \mu(r) > 0,$$

we choose a sequence r_n tending to 0 for which $r_{n+1} < \mu(r_n)$. Then a point of $q_{n+1} = q(r_{n+1})$, other than an end, is interior to D, and [since its distance from $A = r_{n+1} < \mu(r_n)$] is not a point of $D - d_n$, nor (obviously) is it one of q_n. Hence it is interior to d_n. Therefore *q_{n+1} is a cross-cut of d_n*. By (1) it divides d_n into d_{n+1} and another domain Δ_n.

If now P_n is the mid-point of q_n, P_n and P_{n+1} are on free circular arcs of the boundary of Δ_n and can be joined by a simple polygon S_n lying wholly in Δ_n except for the ends. Consider the locus $S_1+S_2+\ldots$. Its moving point can be described by $x = x(\tau)$, $y = y(\tau)$, where τ runs from 0 to $\frac{1}{2}$ on S_1, from $\frac{1}{2}$ to $\frac{3}{4}$ on S_2, and so on. $x(\tau)$, $y(\tau)$ are defined for $0 \leqslant \tau < 1$. But since $m(r_n) \to 0$ we have $x(\tau) \to x_A$, $y(\tau) \to y_A$ as $\tau \to 1$. Completing the definition we have a curve L', $0 \leqslant \tau \leqslant 1$, from P_1 to A.

Since all domains of suffix greater than n are contained in d_{n+1} and therefore have no points common with Δ_n, we see that S_n cannot cut an S of higher suffix. Therefore the curve is simple. We can finally join O and P_1 without meeting L'.

7. *Harmonic functions.*

7.11. In this section we discuss the classical properties of harmonic functions. We take for granted some results belonging to the circle of ideas of Green's theorems and Cauchy's theorem.

The formula of Green is

$$\iint_D \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx\, dy = \int_L (a\, dx + b\, dy) \tag{1}$$

where D is a domain bounded by L, where L consists of a finite number of curves C. If in this formula we take

$$b = \phi \frac{\partial \psi}{\partial x}, \quad a = -\phi \frac{\partial \psi}{\partial y},$$

and write
$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

we obtain

(2)
$$\iint_D \left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y}\right) dx\, dy = -\iint_D \phi \Delta \psi\, dx\, dy + \int_L \phi \left(\frac{\partial \psi}{\partial x} dy - \frac{\partial \psi}{\partial y} dx\right).$$

The three integrands in (2) are invariant for a change of rectangular axes, and the last integral is written, in the usual convention, $\int_L \phi \frac{\partial \psi}{\partial n} ds$, $\frac{\partial}{\partial n}$ denoting differentiation "along the inward normal", and ds the element of arc, taken so that the sense of the axes (dn, ds) is the same as that of Ox, Oy.†

7.12. These formulae have, in any case, no meaning unless the curves C are restricted in some way. We shall always suppose, when there is a question of integrating along a curve C, that C consists of a finite number of portions $x = f_1(t)$, $y = f_2(t)$ $(\alpha \leqslant t \leqslant \beta)$, where f_1' and f_2' exist and are continuous in $\alpha \leqslant t \leqslant \beta$, the various portions being pieced together in such a manner that C is connected. Such a curve C will be called an "elementary curve". It will be found that the apparently ugly restriction does not in the end impair the generality of our conclusions.

The formulae are proved in the first instance when D is bounded, and when respectively a and b have first derivatives continuous in D', or ϕ has continuous first and ψ continuous (first and) second derivatives. In certain important cases, however, we need not assume so much. Suppose that ϕ has first, and ψ second, derivatives continuous at every (internal) point of a bounded D, and that $\Delta \psi = 0$ in D; further that ϕ and ψ have continuous first derivatives in D'. Then

(3)
$$\iint_D \left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y}\right) dx\, dy = \int_L \phi \frac{\partial \psi}{\partial n} ds.$$

† The questions of analysis situs raised by § 7.11 do not involve any fundamental difficulty.

The deduction of this from the original case proceeds on the same lines as the modern proofs of Cauchy's theorem. The result is valid for an L_1 "just inside" L, and the enclosed D_1, and $\iint_{D_1} \to \iint_D$ † as $L_1 \to L$. The proof that $\int_{L_1} \to \int_L$ is the most difficult part of the argument.

This much we take for granted, and also the following: *Suppose that a, b, $\partial a/\partial y$, and $\partial b/\partial x$ are continuous in a* SIMPLY-CONNECTED *domain D. Then if*

$$\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x} \tag{4}$$

for all points (x, y) of D, we have

$$\int_C (a\,dx + b\,dy) = 0 \tag{5}$$

for every closed elementary curve C contained in D, and conversely. Further, if

$$J = \int_{\alpha, \beta}^{x, y} (a\,dx + b\,dy),$$

the integral being taken along any elementary curve in D, then J does not depend on the path from (α, β) to (x, y), and

$$\frac{\partial J}{\partial x} = a, \quad \frac{\partial J}{\partial y} = b.$$

7.21. A function $u(x, y)$ is said to be harmonic in a domain D if it has continuous derivatives of the first two orders, and if $\Delta u = 0$ for all points of D. u is said to be harmonic at a point P if it is harmonic in some neighbourhood of P. A (one valued) function, harmonic at every point of D, is harmonic in D (by Borel's covering theorem).

It is well known that if $f(z)$ is regular at $z = x+iy$, and $f(z) = u+iv$ (where u and v are real), then $\Delta u = \Delta v = 0$. Thus $\mathfrak{R}f(z)$ and $\mathfrak{I}f(z)$ are harmonic in any domain in which $f(z)$ is regular.

We shall sometimes denote, for brevity, the point (x, y) by z, and $u(x, y)$ by $u(z)$. (The reader must, however, guard himself from supposing that u is an analytic function of z.) We shall also sometimes denote (x, y) by its polar coordinates (r, θ) and write $u(r, \theta)$ for $u(x, y)$.

† Practically by *definition* of $\iint_D$.

THEOREM 47. *Let D_1 be a domain bounded by L, a finite set of elementary curves, and let u be harmonic in a D containing D_1'. Then*

$$\int_L \frac{\partial u}{\partial n}\, ds = 0.$$

This is true also if L contains part of the boundary of D, provided that $\partial u/\partial x$, $\partial u/\partial y$ are continuous in D'.

We have only to take $\phi = 1$ in (3) above.

THEOREM 48. *Let u be harmonic in $|z-z_0| < r$ and continuous in $|z-z_0| \leqslant r$. Then*

$$u(x_0, y_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(z_0+re^{\theta i})\, d\theta,$$

Let $$\rho < r,\ I(\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(z_0+\rho e^{\theta i})\, d\theta.$$

We have

(1) $$0 = \int \frac{\partial u}{\partial n}\, ds = -\rho\int_{-\pi}^{\pi} \frac{\partial}{\partial \rho}\, u(z_0+\rho e^{\theta i})\, d\theta.$$

Since $\dfrac{\partial u}{\partial \rho} = \dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\sin\theta$ is continuous in $|z| \leqslant \rho$, $|\theta| \leqslant \pi$, we have

$$\frac{\partial I(\rho)}{\partial \rho} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial u}{\partial \rho}\, d\theta = 0.$$

Hence $I(\rho)$ is constant in $0 \leqslant \rho < r$; also it is evidently continuous in $0 \leqslant \rho \leqslant r$. Hence $I(r) = I(0) = u(z_0)$.

7.22. THEOREM 49 (*Maximum principle*). *Suppose that u is harmonic in a bounded domain D, and that a constant M exists such that, for an arbitrary ϵ and any point P of the boundary $F(D)$,*

$$u < M+\epsilon$$

for all points of D near enough to P. Then $u \leqslant M$ in D, and equality at any point can happen only if u is the constant M. A similar result holds for the opposite inequality.

Let G (possibly $+\infty$) be the upper bound of u in D. It follows by the bisection method that there exists *some* point P of D' such that every neighbourhood S of P gives G as the upper bound of u for points of SD; let us consider the class of all points P of D' with this property. If no P is an interior point, some P is a boundary point. But then the hypothesis of the theorem gives $G < M+\epsilon$, or $G \leqslant M$, while $u = M$ at an interior

point makes that point a P, and is impossible. If some P is an interior point z_0 we have, by the continuity of u, $u(z_0) = G$ (and $G < \infty$). If now C, of radius r, is any circle round z_0 lying, together with its interior, in D, we have, by Theorem 48,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(z_0+re^{\theta i})\,d\theta = u(z_0) = G = \frac{1}{2\pi}\int_{-\pi}^{\pi} G\,d\theta.$$

Now the first integrand does not exceed G. But a *continuous* function whose average is equal to its upper bound is necessarily constant, and it follows that $u = G$ for all points of C. Thus (since r is arbitrary) $u = G$ in the interior of the least circle with P as centre and containing a point of $F(D)$. I say now that $u = G$ throughout D. For if $u < G$ at Q, we join P, Q by a polygon lying in D, and it follows by a Dedekind section that a point R of the polygon exists, such that $u = G$ on the polygon from P to any point short of R, but not from P to any point beyond R. By continuity $u(R) = G$; hence, by the above argument, $u = G$ in some circle round R, and this is contrary to what has just been proved. Thus the hypothesis $u(P) = G$ for an internal P involves $u = G$ throughout D, and then (since $u = G$ near any boundary point) we have $G < M+\epsilon$, or $G \leqslant M$. The statements of the theorem are therefore true in either case.

Another proof proceeds on the lines of Theorem 201 below; this does not require the Dedekind section argument.

THEOREM 50 (*Theorem of uniqueness for given boundary-values*). *Let u_1, u_2 be harmonic in a bounded D, and let them have the same boundary-values. That is, given ϵ and any point P of $F(D)$, then $|u_1-u_2| < \epsilon$ for all points of D near enough to P†. Then $u_1 = u_2$ throughout D.*

$u = u_1-u_2$ is harmonic in D, and satisfies the condition of Theorem 49 with $M = 0$. Hence $u \leqslant 0$ in D, and similarly $u \geqslant 0$.

7.31. Let u be harmonic in D. If a function v exists such that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

throughout D, then v (evidently harmonic in D) is said to be the function conjugate to u. Evidently $v+c$, where c is a constant, is also a conjugate of u. Conversely two conjugates of u differ by a constant (since the

† Observe that two functions can "have the same boundary values" when they are undefined at the actual boundary. The phrase is convenient.

difference w satisfies $w_x = w_y = 0$). If v is conjugate to u, $-u$ is evidently conjugate to v.

THEOREM 51. *Let u be harmonic in a* SIMPLY-CONNECTED† *D. Then there exists a (one-valued) conjugate v, given by*

$$v = \int_{(\alpha,\beta)}^{(x,y)} \left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right).$$

Also $u+iv = f(z)$, a function of the complex variable z regular in D.

The first part follows at once from the last result stated in § 7.12. For the last part we have only to observe that the derivatives u_x, u_y, v_x, v_y are continuous and satisfy the Cauchy-Riemann equations in D, so that $f'(z)$ exists at every point z of D.

It follows from the last part (and § 7.21) that the property of being real and harmonic at a point is *equivalent* to that of being the real part of some function $f(z)$ regular in a neighbourhood of the point.

7.41. THEOREM 52. *Let u and v be any two functions harmonic in a bounded domain D whose boundary L consists of a finite number of elementary curves C, and let u and v have first derivatives continuous in D'. Then*

$$\int_L \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds = 0.$$

For
$$-\int_L = \iint_D (u\Delta v - v\Delta u)\,dx\,dy = 0.$$

An important particular case of the theorem is that in which D is a circular annulus, with centre at, say, the origin. If the circles have radii r_1 and r_2 we have, in fact,

$$\int_{-\pi}^{\pi} \left(u\frac{\partial v}{\partial r} - v\frac{\partial u}{\partial r}\right) r\,d\theta = \textit{constant} \quad (r_1 \leqslant r \leqslant r_2).$$

7.42. Consider now a bounded domain D and one special point of it, which we may suppose without loss of generality to be the origin O. Let $r = (x^2+y^2)^{\frac{1}{2}}$. It is easily verified that $\log r$ is harmonic except at O. Suppose now that $\omega(x, y)$ is harmonic in D and has first differential coefficients continuous in D_1', and let

$$h = h(x, y) = -\log r + \omega(x, y).$$

Suppose, further, that D is bounded by L, consisting of a finite number of elementary curves, and let c be a small circle, of radius ρ,

† If D is multiply-connected the conjugate is many-valued, with moduli of periodicity.

round O. Then, by Theorem 52,

$$(1) \qquad \int_{-\pi}^{\pi}\left(u\frac{\partial h}{\partial r}-h\frac{\partial u}{\partial r}\right)\rho\,d\theta=\int_c=-\int_L\left(u\frac{\partial h}{\partial n}-h\frac{\partial u}{\partial n}\right)ds.$$

Now on c
$$\frac{\partial h}{\partial r}=-\frac{1}{\rho}+\frac{\partial \omega}{\partial r},$$

$$\lim_{\rho\to 0}\rho\int_{-\pi}^{\pi}u\frac{\partial h}{\partial r}\,d\theta=\lim\int_{-\pi}^{\pi}\left(-u+\rho u\frac{\partial \omega}{\partial r}\right)d\theta=-\lim\int_{-\pi}^{\pi}u\,d\theta=-2\pi u(0),$$

and
$$\lim\int_{-\pi}^{\pi}\rho h\frac{\partial u}{\partial n}\,d\theta=0,$$

since $\rho h\to 0$ and $\partial u/\partial n$ is bounded at O. Taking limits in (1) we therefore obtain

$$(2) \qquad u(0)=\frac{1}{2\pi}\int_L\left(u\frac{\partial h}{\partial n}-h\frac{\partial u}{\partial n}\right)ds.$$

Replacing now the origin by the point (x, y), and using ξ, η for current coordinates, we obtain :

THEOREM 53. *Let u be harmonic in D, a domain whose boundary L consists of a finite number of elementary curves, and let u have continuous first derivatives in D'. Then for a point (x, y) of D*

$$u(x, y)=\frac{1}{2\pi}\int_L\left(u\frac{\partial h}{\partial n}-h\frac{\partial u}{\partial n}\right)ds.$$

In particular we may take $h=-\log r$, where r is the distance between (x, y) and (ξ, η).

7.43. *The Green's function of a bounded domain.* This is defined to be the function (if any†) $g(x, y;\, \xi, \eta)$, with the properties (i) $g>0$; (ii) g is harmonic (in ξ, η) for (ξ, η) of D other than (x, y); (iii) near (x, y) $g=-\log r+\omega$, where ω is harmonic for all (ξ, η) of D; (iv) g tends uniformly to 0 as z tends to any point of the boundary.

It is easily shown that (if g exists) $g(x, y;\, \xi, \eta)=g(\xi, \eta;\, x, y)$, but we nowhere make use of this property.

Taking $h=g$ in Theorem 53 we obtain

THEOREM 54. *Suppose that D is bounded by a simple closed elementary curve C, and that g, the Green's function of D, has continuous first derivatives in D' with respect to ξ, η [for every fixed (x, y)]. Let*

† We shall not be concerned here with the existence theorem.

h be the function conjugate to g. Suppose now that (i) *is harmonic in D,* (ii) *u has continuous first derivatives in D′.*† *Then if $U(\xi, \eta)$ is the "boundary value" of u at a point (ξ, η) of C we have*

$$u = \frac{1}{2\pi}\int_C U\frac{\partial g}{\partial n}\,ds = \frac{1}{2\pi}\int_C U\frac{\partial h}{\partial s}\,ds.$$

7 . 44. *The Poisson integral.* It is easy to calculate Green's function for a circle. Let the circle have the origin as centre and radius a. Let P be (x, y), an internal point, P' the inverse point of P with respect to the circle, Q, or (ξ, η), a point within or upon the circumference, ρ and ρ' the distances PQ, $P'Q$. It is a familiar geometrical fact that $\rho/\rho' = k = k(x, y)$ for all Q of the circumference. The function

$$g = -\log\rho + \log\rho' + \log k$$

evidently has the required properties.

A straightforward calculation gives

$$\frac{\partial g}{\partial n} = \frac{a^2-r^2}{a\rho^2} = \frac{1}{a}\,\frac{a^2-r^2}{a^2-2ar\cos(\theta-\psi)+r^2},$$

where (r, θ), (a, ψ) are the polar coordinates of (x, y) and (ξ, η). We obtain now from Theorem 54 the following important result:

THEOREM 55. *Let u be harmonic in a domain containing $r \leqslant a$ $(r^2 = x^2+y^2)$, and let $u(ae^{\psi i}) = U(\psi)$. Then for $r < a$*

(1) $$u = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(\psi)\, P\left(\frac{r}{a},\ \theta-\psi\right) d\psi,$$

where

(2) $$P(\rho, \phi) = \frac{1-\rho^2}{\Delta(\rho, \phi)},\quad \Delta(\rho, \phi) = 1-2\rho\cos\phi+\rho^2.$$

7 . 45. The equation (1) is called Poisson's formula, and u the "Poisson integral of U". We shall have to consider the formula subject to much less restrictive hypotheses, but at the moment we wish to point out other ways by which we may arrive at it, under whatever conditions.

(i) From the point of view in which a harmonic function is the real part of an analytic function $f(z)$ we argue as follows: We may suppose $a = 1$. If $|z_0| < 1$ and we denote by C the circle $z = e^{\psi i}$,

† It is enough, in actual fact, if (ii) is replaced by the continuity of u in D'.

$-\pi \leqslant \psi \leqslant \pi$, we have (supposing f regular in $|z| \leqslant 1$),

(1) $$2\pi f(z_0) = \frac{1}{i}\int_C \frac{f(z)\,dz}{z-z_0} = \int_{-\pi}^{\pi} f(e^{\psi i})\frac{z\,d\psi}{z-z_0} = \int_{-\pi}^{\pi}\{U(\psi)+iV(\psi)\}\frac{e^{\psi i}\,d\psi}{e^{\psi i}-z_0}.$$

Also

$$0 = \frac{1}{i}\int_C \frac{f(z)}{z-1/\bar{z}_0}\,dz = \int_{-\pi}^{\pi} f(e^{\psi i})\frac{\bar{z}_0\,d\psi}{\bar{z}_0-\bar{z}} = \int_{-\pi}^{\pi}(U+iV)\frac{\bar{z}_0\,d\psi}{\bar{z}_0-e^{-\psi i}},$$

since $1/z = \bar{z}$ on C. In this change the sign of i, and of the whole; we obtain

(2) $$0 = \int_{-\pi}^{\pi}(U-iV)\frac{z_0\,d\psi}{e^{\psi i}-z_0}.$$

Adding (1) and (2) we have

(3) $$2\pi\{u(z_0)+iv(z_0)\} = \int_{-\pi}^{\pi}\left(U\frac{e^{\psi i}+z_0}{e^{\psi i}-z_0}+iV\right)d\psi,$$

and, taking real parts,

$$u(z_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} U\,\mathfrak{R}\,\frac{e^{\psi i}+z_0}{e^{\psi i}-z_0}\,d\psi = \frac{1}{2\pi}\int_{-\pi}^{\pi} UP(r_0,\ \theta_0-\psi)\,d\psi.$$

(ii) *Proof by a transformation of Theorem* 47. We shall in time become very familiar with the idea that a property of the *centre* of a circle in which a function is regular or harmonic can generally be asserted, in modified form, of *any* point of the circle. Poisson's formula is such a modification of the formula of Theorem 47.

We must observe first that, if $f(z)$ is regular and $f' \neq 0$ in D, and the equation $\xi+i\eta = \zeta = f(z)$ transforms D (point by point) into a domain Δ of the (ξ, η) plane, then the transformation from (x, y) to (ξ, η) changes a $u(x, y)$, harmonic in (x, y) of D, into a $u_1(\xi, \eta)$, harmonic in (ξ, η), of Δ. Also the conjugate v of u becomes the conjugate v_1 of u_1. We have, in fact, by simple calculations,

$$\xi_x = \eta_y,\quad \xi_y = -\eta_x,\quad \xi_x^2+\eta_x^2 = J = |f'(z)|^2 > 0,$$

$$u_{xx}+u_{yy} = J(u_{\xi\xi}+u_{\eta\eta}),$$

which establish the first part; and for the second we have only to verify that $v_\xi = u_\eta$, $v_\eta = -u_\xi$.

Consider now the transformation†

(4) $$z = \frac{\zeta-\zeta_0}{1-\zeta\bar{\zeta}_0},$$

† This also is destined to become very familiar in the sequel.

where $|\zeta_0| < 1$. This transforms the unit circle of z, boundary included point by point into the unit circle of ζ (and conversely), and $d\zeta/dz$ and $dz/d\zeta$ are never 0 (in either unit circle). Hence an arbitrary $u(x, y)$ harmonic in $|z| \leqslant 1$, transforms into an arbitrary $u_1(\xi, \eta)$ harmonic in $|\zeta| \leqslant 1$.

Take now the formula

$$u(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{\phi i})\, d\phi, \tag{5}$$

and let $z = e^{\phi i}$ and $\zeta = e^{\psi i}$ correspond by (4), and $\zeta_0 = \rho_0 e^{\psi_0 i}$. We have

$$d\phi = \frac{dz}{iz} = -i\,d\log z = -i\left(\frac{d\zeta}{\zeta-\zeta_0} + \frac{\bar{\zeta}_0 d\zeta}{1-\zeta\bar{\zeta}_0}\right) = P(\rho_0, \psi-\psi_0)\, d\psi,$$

and (5) transforms into Poisson's formula for $u_1(\xi_0, \eta_0)$.

7.51. We proceed to develop the theory of the formula subject to more general conditions. We start afresh from a different point of view, suppose in what follows that $U = U(\psi)$ is any function (of period 2π) integrable in the sense of Lebesgue†, and consider the nature of the function

$$u = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(\psi)\, P(r, \psi-\theta)\, d\psi \quad (r < 1), \tag{1}$$

whether the values $U(\psi)$ can be considered as in any sense "boundary-values" of u on $r = 1$.

It is evident that we may differentiate (1) under the integral sign any number of times with respect to x and y, provided $r < 1$. Now it is easily verified that $P(r, \psi-\theta)$ is harmonic in (x, y). Consequently we have

THEOREM 56. *If $U(\psi)$ is of class L, then the function u given by (1) is harmonic in $r < 1$, and has derivatives of all orders, themselves harmonic functions, in $r < 1$.*

7.52. If we take $a = 1$, $u = 1$ in Theorem 55, we obtain

THEOREM 57.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, \theta-\psi)\, d\psi = 1 \quad (r < 1).$$

† It would suffice for our immediate purposes to suppose U continuous, but the extra generality costs us nothing. We shall return to the deeper study of the connexions of u and U in Chapter 3.

This, of course, is incidentally a result in the theory of definite integrals, and may be verified as such.

COR. *If $U(\psi)$ is of class L and $U \leqslant G$ for all ψ, the function u given by* (1) *satisfies $u \leqslant G$.*

Similarly for a lower bound.

For $P > 0$, and so $\int_{-\pi}^{\pi} (U-G)\, P d\psi \leqslant 0.$

7.53. THEOREM 58. *If $U(\psi)$ has period 2π and is of class L, and $\delta < 1$, then*

$$|I(\delta, r, \theta, \theta_0)| = \left|\left(\int_{-\pi}^{\pi} - \int_{\theta_0-\delta}^{\theta_0+\delta}\right) U(\psi)\, P(r, \psi-\theta)\, d\psi\right|$$

$$\leqslant A(\delta)(1-r)\int_{-\pi}^{\pi} |U|\, d\psi \quad (|z-e^{\theta_0 i}| < \tfrac{1}{2}\delta).$$

Thus if $z = re^{\theta i} \to e^{\theta_0 i}$, $I \to 0$ uniformly in θ_0 and in the manner in which z tends to its limit (by a path internal to $r=1$).†

We may suppose that $\theta_0 = 0$. Then $|z-1| < \frac{1}{2}\delta$, $|\theta| < \frac{1}{2}\pi$, and $|\sin\theta| < \frac{1}{2}\delta$. Hence $|\theta| < (1-A)\,\delta$, and (in the ranges over which ψ varies in I) $A\delta < |\psi-\theta| \leqslant \pi$. Then‡

$$\Delta = (1-r)^2 + 4r\sin^2\tfrac{1}{2}(\psi-\theta) > (1-r)^2 + A_1\delta^2 r,$$

or, since this exceeds $\frac{1}{4}$ for $r < \frac{1}{2}$ and $\frac{1}{2}A_1\delta^2$ for $r > \frac{1}{2}$,

$$\Delta > A(\delta), \quad P < (1-r^2)\,A(\delta) < 2(1-r)\,A(\delta),$$

and the desired result follows.

Let $U_+(\psi_0)$ be the limit as $\delta \to 0$ of the upper bound of $U(\psi)$ in $0 < |\psi-\psi_0| \leqslant \delta$, $U_-(\psi_0)$ the corresponding limit of the lower bound. The condition that ψ_0 should be a point of continuity of U is

$$U_+(\psi_0) = U_-(\psi_0) = U(\psi_0).$$

† Thus the behaviour of the Poisson integral of U near $e^{i\psi_0}$ is affected only to the extent of a uniform $o(1)$ by altering the values of U outside an arbitrarily small interval round $\psi = \theta_0$ (so as to leave it of class L). Compare Theorem 33. In fact the Poisson integral of a function U behaves very much like the function $\sigma_n(\theta, U)$, the limit operation $r \to 1$ replacing $n \to \infty$, and the positive kernel P (depending on r), the positive kernel R (depending on n).

‡ It is useful in tentative work to replace the function $P(r, \psi)$ by its effective equivalent

$$\frac{2\eta}{\eta^2+\psi^2},$$

where $\eta = 1-r$ (in fact the two things differ by less than an A).

THEOREM 59. *If $U(\psi)$ is periodic and of class L, and*

$$u = u(r, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(\psi)\, P(r, \psi - \theta)\, d\psi \quad (r < 1),$$

then, uniformly as $z \to e^{\psi_0 i}$ (in $r < 1$),

$$\overline{\lim}\, u \leqslant U_+(\psi_0), \quad \underline{\lim}\, u \geqslant U_-(\psi_0).$$

In particular, if ψ_0 is a point of continuity of U, then

(1) $$u \to U(\psi_0)$$

uniformly as $z \to e^{\psi_0 i}$. If U is continuous in $(-\pi, \pi)$, then (1) *holds uniformly in ψ_0 also, and u is continuous in $r \leqslant 1$.*

Finally the relation "$\lim u = \lim U(\psi)$ *uniformly as* $z \to e^{i\psi_0}$" *holds the limit on the right (as $\psi \to \psi_0$) is $+\infty$ or $-\infty$.*

We have, supposing $U_+(\psi_0)$ finite,

$$U(\psi) < U_+(\psi_0) + \epsilon, \quad \{0 < |\psi - \psi_0| < \delta(\psi_0, \epsilon)\},$$

where we may suppose δ independent of ψ_0 in case U is continuous everywhere.

Let U^* be the function agreeing with U in $|\psi - \psi_0| < \delta$, and having the value $U_+(\psi_0) + \epsilon$ elsewhere, and let u^* be the corresponding Poisson integral.

By Theorem 58 $u - u^*$ tends to 0 as $z \to e^{\psi_0 i}$, and with the maximum degree of uniformity contemplated in the theorem. By Theorem 57, Cor., we have $u^* \leqslant U_+(\psi_0) + \epsilon$, whence

$$\overline{\lim}\, u \leqslant U_+(\psi_0)$$

with the appropriate uniformity. A similar inequality holds for the lower bound, and the two inequalities together establish the desired result.

The cases $U_+(\psi_0) = \pm\infty$ require only obvious modifications.

7 . 54. THEOREM 60. *Suppose that $U(\psi)$ is periodic and continuous in $(-\pi, \pi)$. Then there exists one and only one function u harmonic in $r < 1$, continuous in $r \leqslant 1$, and (for all ψ) taking the value $U(\psi)$ at $z = e^{\psi i}$. This function u is given by the Poisson integral of U.*

This important result is an immediate consequence of Theorems 50, 56, and 59.

7 . 55. THEOREM 61. *Suppose that U is periodic and of class L, and that u is its Poisson integral. Let v be the conjugate of u in $r < 1$.*

Then $$v = v(r, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(\psi)\, Q(r, \psi-\theta)\, d\psi + C,$$

where $$Q = \frac{-2r\sin(\psi-\theta)}{\Delta(r, \psi-\theta)}.$$

For, denoting (r, θ) by z_0 and $e^{\psi i}$ by z,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U\frac{z+z_0}{z-z_0}\, d\psi$$

is differentiable with respect to z_0 under the integral sign. It therefore represents a function of z_0 regular in $r<1$. Since its real part is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U\mathfrak{R}\frac{z+z_0}{z-z_0}\, d\psi = \frac{1}{2\pi}\int_{-\pi}^{\pi} UP\, d\psi = u,$$

it follows that its imaginary part is a conjugate, $v-C$, of u. Since

$$Q = \mathfrak{I}\frac{z+z_0}{z-z_0}$$

(as is easily verified) this proves the theorem.

COR. *Let v denote the particular conjugate* $\frac{1}{2\pi}\int_{-\pi}^{\pi} UQ\, d\psi$. *Then:*

(i) *If* $|U| \leqslant G$ *for all* ψ,

$$|v| \leqslant \frac{2G}{\pi}\log\frac{1+r}{1-r}.$$

(ii) *In any case*

$$|u| \leqslant \frac{1+r}{1-r}M_1, \quad |v| \leqslant \frac{2r}{1-r}M_1,$$

where $$M_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|U|\, d\psi.$$

In fact, we have in case (i)

$$|v| \leqslant \frac{G}{2\pi}\int_{-\pi}^{\pi}|Q|\, d\psi \leqslant \frac{G}{\pi}\int_0^{\pi}\frac{2r\sin\phi\, d\phi}{\Delta(\phi)} = \frac{G}{\pi}\Big[\log\Delta(\phi)\Big]_0^{\pi} = \frac{2G}{\pi}\log\frac{1+r}{1-r},$$

and, in case (ii),

$$P \leqslant \frac{1-r^2}{(1-r)^2} = \frac{1+r}{1-r}, \quad |Q| \leqslant \underset{(\psi)}{\mathrm{Max}}\left|\mathfrak{I}\frac{z+z_0}{z-z_0}\right| = \frac{2r}{1-r},$$

whence we obtain the required results.

7.6. *General theorems.*

7.61. We now apply the results of 7.5 to the general theory of harmonic functions.

THEOREM 62. *A function harmonic in a domain D has in D partial derivatives of all orders, themselves harmonic functions.*

Any point of D has a small circular neighbourhood also belonging (when taken closed) to D; by Theorem 55 u is equal to a Poisson integral inside the circle, and, by Theorem 56, has harmonic derivatives of all orders there.

7.62. THEOREM 63. *Let the functions u_n be harmonic in D and continuous in D', and suppose that the boundary values U_n of u_n converge uniformly to some boundary function U. Then u_n tends uniformly in D to a function u, harmonic in D, continuous in D', and with boundary function U. Also any derivative of u_n converges to the corresponding derivative of u, uniformly in any closed set contained in D. Similar results hold for a $u(x, y, t)$ depending on a continuous parameter t in place of n.*

We have $|U_m - U_n| < \epsilon$ $(m, n > n_0)$. Hence, by Theorem 49, $|u_n - u_m| < \epsilon$ for $m, n > n_0$ and all points of D, and u_n tends to a limit function u uniformly in D.

Consider now any circle C in D, and let the values of u_n and u on C be denoted by $U_n(C)$ and $U(C)$. Then

$$U_n(C) = U(C) + o(1),$$

o's being uniform in the variable point of C. Inside C we have

$$u_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} U_n(C) P\, d\psi \quad \text{(Theorem 55)}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} U(C) P\, d\psi + \vartheta \frac{1}{2\pi}\int_{-\pi}^{\pi} |U_n(C) - U(C)|\, P\, d\psi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} U(C) P\, d\psi + o(1) \quad \text{(Theorem 56).}$$

It follows that

$$u = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(C) P\, d\psi,$$

and hence that u is harmonic.

Let now C have radius r, and consider C_1, the concentric circle of smaller radius ρ. We have in C_1, denoting by D either $\dfrac{\partial}{\partial x}$ or $\dfrac{\partial}{\partial y}$,

$$(1) \qquad Du_n = \frac{1}{2\pi}\int U_n(C)DP\,d\psi, \quad Du = \frac{1}{2\pi}\int U(C)DP\,d\psi.$$

Now DP is continuous in x, y, ψ if (x, y) is in or upon C_1 and ψ is the amplitude of the variable point of C; thus

$$|DP| < A(r, \rho)$$

in the integrals in (1). Hence

$$|Du_n - Du| \leqslant \frac{1}{2\pi}\int |U_n(C) - U(C)|\,|DP|\,d\psi = o(1),$$

and $Du_n \to Du$ uniformly in C_1. *Since Du_n, Du are harmonic* (so that the argument can be repeated), and since any closed set interior to D can be covered† by a finite number of circles C_1, the proof is completed.

COR. *Suppose (in addition) that D is simply-connected, and that v_n is a conjugate of u_n and v a conjugate of u. Then there exist constants c_n such that $v_n - c_n \to v$ uniformly in any closed set contained in D.*

Let

$$(2) \qquad v_n{}^* = \int_{(\alpha,\beta)}^{(x,y)} \left(-\frac{\partial u_n}{\partial y}dx + \frac{\partial u_n}{\partial x}dy\right),$$

which (by § 7 . 12) is independent of the *path*. Given any domain D_1 such that $D_1' \subset D$, D_1 can be covered by a finite set of circles each contained in D, and there exists a K such that any two points of D_1 can be joined by a rectilinear path lying within the sum of the circles and of length not exceeding K. Since $Du_n \to Du$ uniformly in D_1, it follows from (2) that

$$v_n{}^* \to v^* = \int_{(\alpha,\beta)}^{(x,y)} \left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right)$$

uniformly in D_1, and v_n^* and v^* are particular conjugates of u_n and u.

7 . 63. THEOREM 64. *Let $u_1, u_2, \ldots$ be an increasing sequence of functions harmonic in D. If now u_n converges at one point A of D it converges uniformly to a function u in any closed set contained in D, where u is harmonic in D.*

† In virtue of the theorem of Borel.

Evidently $$u_n = \sum_1^n w_n,$$

where w_n is, for $n > 1$, a non-negative harmonic function.

Let C be a circle about A (which we may suppose to be the origin), interior to D and of radius a, and let $\rho < a$. In a circle of radius $r \leqslant \rho$ we have

$$w_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} W_n(C) P\left(\frac{r}{a},\ \psi - \theta\right) d\psi$$

(in the notation of § 7 . 62). Now

$$0 \leqslant P \leqslant \frac{a+\rho}{a-\rho},$$

whence, if $n > 1$, and so $W_n \geqslant 0$,

$$w_n \leqslant \frac{a+\rho}{a-\rho}\frac{1}{2\pi}\int_{-\pi}^{\pi} W_n(C)\, d\psi = \frac{a+\rho}{a-\rho}\, w_n(A).$$

Since $\sum_2^\infty w_n(A)$ is a convergent series of non-negative terms, it follows that Σw_n converges uniformly in $r \leqslant \rho$.

It is easy to extend the region of uniform convergence to any D_1 for which $D_1' \subset D$. If Σw_n converges uniformly in a neighbourhood of each point of D_1' it converges uniformly in D_1', by Borel's "covering theorem" (a finite number of neighbourhoods can be made to cover D_1'). If now there is not uniform convergence at † every point of D_1, consider the set E of points of non-uniform convergence ‡, and let B be the point of E' nearest to A (evidently $BA \geqslant \rho$). Clearly there exists a circle C', with centre at a point P of uniform convergence, lying wholly in D, and containing B in its interior. Since Σw_n converges at P it converges uniformly in a circle just smaller than C', and so converges uniformly in a neighbourhood of B, and, again, in a neighbourhood of every point of E near enough to B. This gives a contradiction and completes our proof.

7 . 64. THEOREM 65. (*Analogue of Liouville's theorem.*) *A bounded function harmonic everywhere is necessarily a constant.*

It is sufficient to prove that $u(r, \theta) = u(0)$. If C is $|z| = R > r$, we

† Uniform convergence "at" P means, of course, uniform convergence in some neighbourhood of P.

‡ By this we mean, of course, to include points of non-convergence.

have, supposing $|u| \leqslant K$,

$$2\pi\{u(r,\theta)-u(0)\} = \int_{-\pi}^{\pi} u(R,\psi)P\left(\frac{r}{R},\ \psi-\theta\right)d\psi - \int_{-\pi}^{\pi} u(R,\psi)\,d\psi,$$

$$2\pi\,|u(r,\theta)-u(0)| \leqslant \int_{-\pi}^{\pi}\left|P\left(\frac{r}{R},\psi-\theta\right)-1\right| K\,d\psi$$

$$= K\int_{-\pi}^{\pi}\frac{|\{R^2-r^2\}-\{R^2-2Rr\cos(\psi-\theta)+r^2\}|}{R^2-2Rr\cos(\psi-\theta)+r^2}\,d\psi \leqslant K\,2\pi\,\frac{2Rr+2r^2}{(R-r)^2}.$$

Since this tends to zero as $R \to \infty$ the left-hand side must be zero.

7 . 7. *Functions harmonic in a circle or in an annulus.*

7 . 71. THEOREM 66. (*Analogue of Taylor's theorem.*) *Suppose that u is harmonic in $r < R$ and v is a conjugate of u. Then*

(1)
$$\left.\begin{aligned} u &= \tfrac{1}{2}a_0 + \sum_1^\infty (a_n\cos n\theta + b_n \sin n\theta)\,r^n \\ v+c &= \sum_1^\infty (-b_n\cos n\theta + a_n\sin n\theta)\,r^n \end{aligned}\right\} \quad (r < R),$$

where

(2) $$a_n = \frac{1}{\pi R_1^n}\int_{-\pi}^{\pi} u(R_1 e^{\psi i})\cos n\psi\,d\psi, \quad b_n = \frac{1}{\pi R_1^n}\int_{-\pi}^{\pi} u(R_1 e^{\psi i})\sin n\psi\,d\psi,$$

for every R_1 satisfying $0 < R_1 < R$: The series

(3) $$\Sigma(|a_n|+|b_n|)\,r^n$$

is absolutely convergent for $r < R$, and (1) *is uniformly convergent in $r \leqslant R-\delta$ and all θ. Further, any expansion of u or of $v+c$ in a series of the form in* (1) *which converges uniformly*† *on (say) $r = r_0$, necessarily has coefficients given by* (2). *Conversely, if the a's and b's are any numbers such that* (3) *converges for $r < R$, then the right-hand sides in* (1) *are conjugate harmonic functions in $r < R$.*

Let $R_2 < R_1 < R$. By Theorem 55 we have in $r \leqslant R_2$

(4) $$u = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(R_1 e^{\psi i})\,P\left(\frac{r}{R_1},\ \psi-\theta\right)d\psi.$$

Since

$$P\left(\frac{r}{R_1},\ \psi-\theta\right) = 1 + 2\left(\frac{r}{R_1}\right)^n \cos n(\psi-\theta),$$

a series uniformly convergent in $r \leqslant R_2$ and all ψ, θ, we may integrate term by term in (4), obtaining the first equation in (1) for $r \leqslant R_2$.

† We cannot, as we can for a power series, infer the uniform convergence for $r \leqslant r_1 - \delta$ from the convergence at a *single* point on $r = r_1$.

Since $$= 2\Sigma \left(\frac{r}{R_1}\right)^n \sin n(\psi - \theta)$$

the second equation is proved similarly.

Since R_2 may be arbitrarily near R the first part of the theorem is proved. The second part is a consequence of

$$|a_n| + |b_n| \leqslant \frac{2}{\pi R_1^n} \int_{-\pi}^{\pi} |u(R_1 e^{\psi i})| \, d\psi < K(R_1) \,.\, R_1^{-n}.$$

For the uniqueness of the series developments, suppose that

$$u(r_0, \theta) = \tfrac{1}{2}a_0' + \sum_1^\infty (a_n' \cos n\theta + b_n' \sin n\theta)\, r_0^n,$$

where the right-hand side converges uniformly (in θ). Then

$$\tfrac{1}{2}(a_0' - a_0) + \sum_1^\infty \{(a_n' - a_n)\cos n\theta + (b_n' - b_n)\sin n\theta\}\, r_0^n$$

converges, uniformly in θ, to 0. We may now multiply by $\cos n\theta$ or and integrate term by term between $-\pi$ and π, and this gives

$$a_n' - a_n = b_n' - b_n = 0.$$

The uniqueness result, combined with what has been proved, shows now that the R_1 in (2) is arbitrary.

For the converse we observe that $\Sigma n^r(|a_n| + |b_n|)(R-\delta)^n$ is convergent, so that (for $r < R$) we may differentiate any number of times under the summation signs in (1). It follows that u, v have continuous derivatives of all orders, and, since $r^n \cos n\theta$ and $r^n \sin n\theta$ are harmonic, that $\Delta u = \Delta v = 0$ and u, v satisfy the differential equations connecting conjugate functions. This completes the proof.

COR. *If, in particular, $R = 1$ and u is the Poisson integral of a function $U(\psi)$ (of class L), then a_n, b_n are the Fourier coefficients of U. If a u given by* (1) *has a continuous boundary function U, then a_n, b_n are the Fourier coefficients of U.*

If U is continuous and a_n, b_n are its Fourier coefficients, then the u given by (1) *is harmonic in $r < 1$, continuous in $r \leqslant 1$, and has U as boundary function.*

In particular, the Abel limit of the Fourier series of a continuous function U exists uniformly and is equal to U.

For in $$u = \frac{1}{2\pi} \int UP\,d\psi$$

we may expand P in a series

(5) $$1 + 2\Sigma r^n \cos n(\psi - \theta),$$

uniformly convergent in $r \leq 1-\delta$, and integrate term by term. This gives

(6) $$u = \tfrac{1}{2}\alpha_0 + \sum_1^\infty (\alpha_n \cos n\theta + \beta_n \sin n\theta)\, r^n,$$

where

(7) $$\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} U \cos n\psi\, d\psi, \quad \beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} U \sin n\psi\, d\psi,$$

and the series (6) is, like (5), uniformly convergent in $r \leqslant 1-\delta$. By the uniqueness of the series development, $\alpha_n = a_n$, $\beta_n = b_n$; and, by (7), a_n, β_n are the Fourier coefficients of U.

The last parts are immediate, since u is the Poisson integral of U (Theorem 60).

7.72. From Theorem 66 we can deduce the following result in general theory.

THEOREM 67. *Suppose that u_1, u_2 are harmonic in D, and that $u_1 = u_2$ in a domain*† *$D_1 \subset D$. Then $u_1 = u_2$ throughout D.*

If the theorem is false it is evidently possible to find concentric circles C_1, C_2, with radii r_1 and $r_2 > r_1$, both entirely interior to D, and such that $u = u_1 - u_2 = 0$ in and upon C_1, but not everywhere within C_2. Taking the origin at the common centre we have a series development for u within C_2. But the coefficients of this series are given by (2) of Theorem 66, and the integrands are identically zero if $0 < R_1 < r_1$. Thus the coefficients are zero, and $u = 0$ in C_2, which gives a contradiction.

7.73. THEOREM 68. (*Analogue of Laurent's theorem.*) *Suppose that $0 < r_1 < r_2$, and that u is harmonic in a domain containing the annulus $r_1 \leqslant r \leqslant r_2$. Then for an (x, y) of $r_1 < r < r_2$,*

(1) $$u = u_i + u_e,$$

where

(2) $$u_i(x, y) = -\frac{1}{2\pi}\int_{C_2}\left(u(\xi, \eta)\frac{\partial \log R}{\partial n} - \log R\,\frac{\partial u(\xi, \eta)}{\partial n}\right) ds,$$

(3) $$u_e(x, y) = -\frac{1}{2\pi}\int_{C_1}\left(u(\xi, \eta)\frac{\partial \log R}{\partial n} - \log R\,\frac{\partial u(\xi, \eta)}{\partial n}\right) ds,$$

C_1 and C_2 are $r = r_1$ and $r = r_2$ respectively, and

$$R = \sqrt{\{(\xi - x)^2 + (\eta - y)^2\}}.$$

† In the familiar analogue for analytic functions $f(z)$ it is enough to suppose that $f_1 = f_2$ along a curve, or indeed merely at an infinity of points with a limit point interior to D. These extensions are false for harmonic functions.

Further :

(4) *u_i is harmonic in $r < r_2$, u_e harmonic in $r > r_1$;*

(5) *in* (2) *and* (3) *we may replace C_1 by any circle C'_1 between C_1 and (x, y), and C_2 by any circle C'_2 between C_2 and (x, y);*

(6) *u_i is of the form*

$$a_0 + \sum^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\, r^n \qquad (r < r_2),$$

where $\Sigma(|a_n| + |b_n|)\, r^n$ converges for $r < r_2$, so that the series for u_i converges uniformly for $r \leqslant r_2 - \delta$;

(7) *u_e is of the form*

$$k \log r + \sum_1^{\infty} (a'_n \cos n\theta + b'_n \sin n\theta)\, r^{-n} \qquad (r > r_1),$$

where $\Sigma(|a'_n| + |b'_n|)\, r^{-n}$ converges for $r > r_1$, so that the series for u_e converges uniformly for $r \geqslant r_1 + \delta$;

(8) *the results* (1), (4), (6), (7) *are valid also if u is harmonic only in $r_1 < r < r_2$.*

Finally

(9) *Any two expansions for u of the type*

$$\text{(10)} \qquad \sum_1^{\infty} (a_n \cos n\theta + b_n \sin n\theta)\, r^n + k \log r + \sum_1^{\infty} (a'_n \cos n\theta + b'_n \sin n\theta)\, r^{-n}$$

are identical, provided that on two distinct circles $r = r_0$, $r = r'_0$ of the annulus they represent u and are uniformly convergent.

The result expressed by (1), (2), and (3) is an immediate consequence of Theorem 53. Next, we may differentiate any number of times under the integral sign in (2) if $r < r_2$, and in (3) if $r > r_1$. Since the integrands are harmonic in (x, y) in these domains we obtain the result (4). The two results in (5) follow from the special case of Theorem 52 given at the end of § 7 . 41.

(6) is a consequence of (4) and Theorem 66; but (7) is less immediate. We may write (3) in the form

$$\text{(11)} \qquad -2\pi u_e = \int_{C_1} u \frac{\partial \log R}{\partial n}\, ds + \log r \int_{C_1} \frac{\partial u}{\partial n}\, ds - \int_{C_1} \log \frac{R}{r} \frac{\partial u}{\partial n}\, ds.$$

Now for points (ξ, η) or (r_1, ψ) of C_1, $\dfrac{\partial \log R}{\partial n} \left(= \dfrac{1}{R} \dfrac{\partial R}{\partial n} \right)$ and $\log \dfrac{R}{r}$ may

each be expanded in the form $\Sigma c_n r^{-n} \cos n(\psi-\theta)$, where $\Sigma |c_n|(r_1+\delta)^{-n}$ is convergent.† Since u and $\frac{\partial u}{\partial n}$ are continuous on C_1 we may integrate term by term in (11), obtaining the various results in (7).

Next consider (9), still supposing u harmonic in a domain containing $r_1 \leqslant r \leqslant r_2$. In virtue of the uniform convergence we may multiply the expansion for u by $\cos n\theta$ or $\sin n\theta$ and integrate term by term on either $r = r_0$, or $r = r_0'$. With $\cos n\theta$ and $n > 0$ this evidently gives two equations which determine a_n and a_n', and the case $n = 0$ gives two equations to determine k and a_0. Similarly with $\sin n\theta$ we determine b_n and b_n' (there being, of course, no exceptional case $n = 0$).

Finally it is evident, *in virtue of* (5) *and* (9), that we need suppose u harmonic only in $r_1 < r < r_2$ to secure the truth of (1), (4), (6), and (7).

COR. *If u is (one-valued and) harmonic in $0 < r < r_2$ then*

$$u = u_i + u_e,$$

where u_i is harmonic in $r < r_2$, u_e is harmonic in $r < 0$, and

(13) $$u_i = a_0 + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta)\, r^n \quad (r < r_2),$$

(14) $$u_e = k \log r + \sum_1^\infty (a_n' \cos n\theta + b_n' \sin n\theta)\, r^{-n} \quad (r > 0),$$

where $\Sigma(|a_n| + |b_n|)\, r^n$ converges for $r < r_2$ and $\Sigma(|a_n'| + |b_n'|)\, r^{-n}$ converges for all positive r.

7.74. THEOREM 69. (*Analogue of Osgood's theorem.*) *Suppose that u is (one-valued and) harmonic in a neighbourhood of a point P, except at P itself, and that u is bounded in the neighbourhood of P. Then u is harmonic also at P.*

We may suppose P to be the origin and the neighbourhood to contain $r \leqslant r_2$. By Theorem 68, Cor., and with its notation we have, supposing K to be a bound of $|u|$ in $r \leqslant r_2$, and integrating term by term,

$$|k \log r + a_0| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} u(r, \theta)\, d\theta\right| \leqslant K.$$

† If $z_0 = r_1 e^{\psi i}$, $z = re^{\theta i}$, then

$$\frac{\partial \log R}{\partial n} = \mathfrak{R}\, z_0 \frac{\partial}{\partial z_0} \log(z - z_0) = \mathfrak{R} \sum_1^\infty \left(\frac{z}{z_0}\right)^{-n} = \sum_1 \left(\frac{r}{r_1}\right)^{-n} \cos n(\psi - \theta);$$

$$\log \frac{R}{r} = \mathfrak{R} \log \frac{z - z_0}{z} = -\sum_1^\infty \frac{1}{n} \left(\frac{r}{r_1}\right)^{-n} \cos n(\psi - \theta).$$

This being true for arbitrarily small r we have $k = 0$. Similarly, for $n > 0$,

$$|a_n' r^{-n} + a_n r^n| = \left| \frac{1}{\pi} \int_{-\pi}^{\pi} u(r, \theta) \cos n\theta \, d\theta \right| \leqslant \frac{1}{\pi} \int_{-\pi}^{\pi} K \, d\theta = 2K.$$

Since this is true for arbitrarily small r we have $a_n' = 0$, and similarly $b_n' = 0$. Hence

$$u = a_0 + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta) r^n$$

for $0 < r < r_2$, and so for $0 \leqslant r < r_2$ if we define $u(0) = a_0$.

Since $\Sigma(|a_n| + |b_n|) r^n$ converges for $r < r_2$ u is harmonic in $r < r_2$, by Theorem 66 (converse part).

7 . 75. THEOREM 70. *Suppose that the real function u is (one-valued and) harmonic in the neighbourhood of a point P, except at P, and that $u \to +\infty$ as (x, y) tends to P. Then $u = k \log r + u_1$, where k is negative and u_1 is harmonic at P. A similar result holds if $u \to -\infty$, k being positive.*

We may suppose P to be the origin, and in the notation of Theorem 68, Cor., we have to prove that $a_n' = b_n' = 0$. Now if, *e.g.*, $u \to +\infty$ we have, for any large positive h,

$$u > h \quad (0 < r < r_0).$$

Hence, for $n > 0$ and $0 < r < r_0$,

$$\int_{-\pi}^{\pi} u \left(1 \pm {\cos \atop \sin} n\theta\right) d\theta \geqslant \int_{-\pi}^{\pi} h \left(1 \pm {\cos \atop \sin} n\theta\right) d\theta = 2\pi h,$$

since the factor in brackets is non-negative. Hence we have

$$2\pi(a_0 + k \log r) \pm \pi(a_n' r^{-n} + a_n r^n) = \int_{-\pi}^{\pi} u(1 \pm \cos n\theta) \, d\theta \to +\infty$$

as $r \to 0$, *whichever sign is taken*. This clearly requires $a_n' = 0$ (and $k < 0$). Similarly $b_n' = 0$.

Theorem 70 is the analogue of the theorem that if $|f(z)| \to \infty$ at an isolated uniform singularity, then that singularity must be a pole.

7 . 76. We conclude by giving the existence theorem for a circular annulus with assigned continuous boundary values.

THEOREM 71. *Suppose that $0 < r_1 < r_2$, and let us denote the open and closed annuli $r_1 < r < r_2$, $r_1 \leqslant r \leqslant r_2$ by D and D'. Suppose, further, that $U_1(\theta)$, $U_2(\theta)$ are continuous and periodic functions of θ.*

Then there exists one and only one function u harmonic in D, continuous in D', and such that

$$u(r_1, \theta) = U_1(\theta), \quad u(r_2, \theta) = U_2(\theta).$$

Further, if

$$U_i \sim \Sigma(a_{n,i} \cos n\theta + b_{n,i} \sin n\theta) \quad (i = 1, 2),$$

then we have in D,

(1) $$u = k \log r + \tfrac{1}{2}\alpha_0 + \sum_1^\infty (\alpha_n \cos n\theta + \beta_n \sin n\theta) r^n$$
$$+ \sum_1^\infty (\gamma_n \cos n\theta + \delta_n \sin n\theta) r^{-n}$$
$$= k \log r + \tfrac{1}{2}\alpha_0 + u_1 + u_2,$$

where the coefficients k, α, β, γ, δ are so chosen that the series (1) *becomes formally identical with the Fourier series of U_i when $r = r_i$ $(i = 1, 2)$. Finally*

(2) $$\Sigma(|\alpha_n| + |\beta_n|) r^n, \quad \Sigma(|\gamma_n| + |\delta_n|) r^{-n}$$

respectively convergent for $r < r_2$, $r > r_1$, and (1) *is uniformly convergent for $r_1 + \delta \leqslant r \leqslant r_2 - \delta$ and all θ.*

The conditions of formal identity just determine the coefficients in (1), and in virtue of

$$a_{n,i} = O(1), \quad b_{n,i} = O(1)$$

(as $n \to \infty$) the explicit formulae give at once the results about the convergence of the series (2). It follows that the series for u_1 with $r = r_2$ is the Fourier series, $\Sigma(a^*_{n,2} \cos n\theta + b^*_{n,2} \sin n\theta)$, say, of

$$U_2^*(\theta) = U_2(\theta) - (k \log r_2 + \tfrac{1}{2}\alpha_0) - u_1(r_2, \theta).$$

Since this function is continuous

$$u_1(\lambda r_2, \theta) = \Sigma(a^*_{n,2} \cos n\theta + b^*_{n,2} \sin n\theta) \lambda^n$$

tends uniformly to U_2^* as $\lambda \to 1-0$ (Theorem 66, Cor., last part). Quite similarly, with an obvious notation,

$$u_2(\lambda^{-1} r_1, \theta) = \Sigma(a^*_{n,1} \cos n\theta + b^*_{n,1} \sin n\theta) \lambda^n$$

tends uniformly to U_1^* as $\lambda \to 1-0$. It follows easily [since the series for $u_2(r_2, \theta)$, $u_1(r_1, \theta)$ are crudely convergent] that

(3) $$u(r, \theta; \lambda) = k \log r + \tfrac{1}{2}\alpha_0 + u_1(\lambda r, \theta) + u_2(\lambda^{-1} r, \theta)$$

tends uniformly to U_1 when $r=r_1$ and $\lambda\to 1$, and to U_2 when $r=r_2$ and $\lambda\to 1$. It follows from Theorem 63 that as $\lambda\to 1$ $u(r,\theta;\lambda)$ tends uniformly in D' to a function u^*, harmonic in D and continuous in D', and with U_1, U_2 as boundary values. If finally $r_1<r<r_2$, (3) gives

$$u^*=\lim u(r,\theta;\lambda)=k\log r+\tfrac{1}{2}a_0+u_1(r,\theta)+u_2(r,\theta),$$

and this completes the proof.

8. *The behaviour of certain special functions of a complex variable.*

8.1. In much of our subsequent work it is of great value to possess examples of all the important varieties of asymptotic behaviour. The behaviour of each of the functions discussed in this section is in some respect extreme. Our functions

$$f(z)=\Sigma a_n z^n$$

are all regular in the unit circle γ. The mean $M_\lambda\{f(\rho e^{\theta i})\}$ we denote by $M_\lambda(\rho,f)$, or by $M_\lambda(\rho)$. Thus

$$M_\lambda(\rho,f)=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\rho e^{\theta i})|^\lambda d\theta\right)^{1/\lambda}\quad(\lambda>0).$$

We write $F(\rho)$ for the majorant $\Sigma|a_n|\rho^n$ of $f(z)$.

8.2. The first function we consider is

$$f(z)=\Sigma n^{-2}z^{2^n}.$$

This is evidently continuous in $|z|\leqslant 1$, and it has the property that $\Sigma|a_n|^\lambda n^\epsilon$ diverges for all positive λ and ϵ. We describe $f(z)$, for obvious reasons, as a "gap-function".

8.3. Consider next the unbounded gap-function

$$f(z)=\Sigma n!\,e^{i\alpha_n}z^{n!}.$$

Let $u_n=n!\,e^{i\alpha_n}z^{n!}$, $\rho_n=\exp(-1/n!)$, and consider f first when $\rho=\rho_n$. The function $\phi(u)=u\rho^u$ is a maximum (in $u\geqslant 0$) when $u=u_0=1/\log 1/\rho$, and $\phi(u_0)=e^{-1}u_0$. It follows that $|u_m|$ is greatest when $m=n$. We have, in fact,

$$\left|\frac{u_{n-s}}{u_n}\right|<\rho_n^{-n!}\frac{1}{n(n-1)\dots(n-s+1)}=\frac{e}{n(n-1)\dots(n-s+1)}\quad(s=1,2,\dots,n),$$

$$\left|\frac{u_{n+s}}{u_n}\right|=e(n+1)(n+2)\dots(n+s)\,e^{-(n+1)\dots(n+s)}<e(n+s)\,e^{-(n+s)}<Ae^{-\frac{1}{2}(n+s)}$$
$$(s=1,2,\dots).$$

Hence

$$\left|\frac{f(z)}{u_n}\right| = 1+\vartheta\left\{e\left(\frac{1}{n}+\frac{1}{n(n-1)}+\ldots+\frac{1}{n!}\right)+Ae^{-\frac{1}{2}n}(1+e^{-\frac{1}{2}}+e^{-1}+\ldots)\right\}$$

$$= 1+\frac{A\vartheta}{n},$$

$$|f(z)| \sim |u_n| = e^{-1}\left(\log\frac{1}{\rho}\right)^{-1} \sim \frac{e^{-1}}{1-\rho} \quad (\rho=\rho_n)$$

uniformly in θ and the a_n's. Thus $F(\rho)$, $M(\rho) = \underset{|z|=|\rho}{\text{Max}}|f|$, and $M_\lambda(\rho)$ are all asymptotically equivalent to $e^{-1}(1-\rho)^{-1}$ as $\rho \to 1$ through the ρ_n. On the special circles $\rho = \rho_n$, in fact, there is complete dominance of the series $\Sigma a_n z^n$ by a single term.

It is easily seen that for other values of ρ f is dominated by at most two terms. Suppose, in fact, that $\rho_n \leqslant \rho < \rho_{n+1}$, and of the two terms u_n, u_{n+1} let u_n, say, have the greater modulus†: $|u_{n+1}| = \sigma|u_n|$, $0 < \sigma \leqslant 1$. Then we can show, much as above, that

$$\left(\sum_{m<n}+\sum_{m>n+1}\right)|u_m| < \frac{A}{n}|u_n|,$$

whence

(1) $$|f| = |u_n|\,|1+\sigma e^{i\gamma+\nu\theta i}|+o(|u_n|),$$

where $\nu = (n+1)!-n!$. From (1) we have

(2) $$M(\rho) = \{1+\sigma+o(1)\}|u_n|, \quad F(\rho) = \{1+\sigma+o(1)\}|u_n|,$$

$$M_\lambda(\rho) = |u_n|\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|1+\sigma e^{i\gamma+\nu\theta i}|^\lambda\, d\theta\right)^{1/\lambda}+o(|u_n|).$$

By change of variable from θ to θ/ν the coefficient of $|u_n|$ is

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|1+\sigma e^{i\gamma+\theta i}|^\lambda\, d\theta\right)^{1/\lambda} = M_\lambda(\phi),$$

where $\phi = 1+\sigma e^{\theta i}$. Now if $\sigma < 1$,

$$\log M_0(\phi) = \Re\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\log(1+\sigma e^{\theta i})\, d\theta = \Re\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\Sigma\frac{(-)^{n-1}}{n}\sigma^n e^{n\theta i}\, d\theta = 0,$$

and so [Theorem 1, (10)] $M_\lambda(\phi) \geqslant M_0(\phi) = 1$ $(\lambda > 0)$. Since $M_\lambda(\phi)$ is

† Since in any case $|u_n| \geqslant e^{-1}n!$ it follows that the greater term is *large*.

continuous in σ for $\sigma \leqslant 1$ when $\lambda > 0$ we have also $M_\lambda(\phi) \geqslant 1$ for $\sigma \leqslant 1$†. Hence, finally

(3) $$M_\lambda(\rho) > \{1+o(1)\}\,|u_n| \quad (\lambda > 0).$$

It follows from (2) that

(4) $$M_\lambda(\rho) > (\tfrac{1}{2}-\epsilon)\,M(\rho), \quad M(\rho) \sim F(\rho).$$

A precisely similar argument shows that the results (4) hold also if $|u_{n+1}| > |u_n|$. They hold therefore as $\rho \to 1$ through all values.

8.4. Our next function is

$$f(z) = (1-z)^{-\alpha}\left\{\frac{1}{z}\log\frac{1}{1-z}\right\}^{\beta}$$

where $\alpha \geqslant 0$, and $\beta > 0$ if $\alpha = 0$, so that f is unbounded in γ.

We write B for $A\ (\alpha, \beta)$. We shall show first that, as $n \to \infty$,

(1) $$a_n \sim Bn^{\alpha-1}(\log n)^{\beta} \quad (\alpha > 0),$$

(2) $$a_n \sim Bn^{-1}(\log n)^{\beta-1} \quad (\alpha = 0,\ \beta > 0).$$

It is enough to prove (1), since if $\alpha = 0$,

$$\Sigma(n+\beta)\,a_n z^n = zf'(z)+\beta f(z) = \beta(1-z)^{-1}\left\{\frac{1}{z}\log\frac{1}{1-z}\right\}^{\beta-1}$$

and $na_n \sim (n+\beta)\,a_n \sim B(\log n)^{\beta-1}$ by (1).

Similarly, by repeated differentiation, it is enough to prove (1) when α is greater than any convenient constant, in particular when $\alpha > |\beta|+1$. Let now

$$r = e^{-1/n}, \quad f^* = (1-z)^{-\alpha}\left\{\frac{1}{r}\log\frac{1}{1-r}\right\}^{\beta}.$$

We have $$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z^{n+1}},$$

where C is the circle $|z| = r$. Our argument is, roughly, that f and f^* differ trivially on a certain part of C of length large compared with $1-r$ (but not too large), while the contribution of the rest of C to a_n is negligible. This reduces the problem to the simpler case $\beta = 0$. As it is on these grounds that the results are intuitive to the expert, a proof based on them is perhaps the right one; the details, however, seem a little awkward (and the experienced reader will omit them).

† This also follows at once from Theorem 208 below, in virtue of which $M_\lambda(\rho, \varphi)$ is an increasing function of ρ, and so $M_\lambda(\rho) \geqslant M_\lambda(0)$.

Let C_1 be the part of C for which $|\theta| \leqslant (1-r)\log\{(1-r)^{-1}\} = \eta$, C_2 the remainder. On C_1

(3) $$1-z = 1-r+i\theta+O\{\theta(1-r)\}+O(\theta^2),$$

$$\frac{1-z}{1-r} = 1+i\frac{\theta}{1-r}+o(1),$$

$$\log\frac{1-z}{1-r} = \begin{cases} O(1) \text{ if } |\theta| < 2(1-r) \\ O\left(\log\log\frac{1}{1-r}\right) \text{ if } 2(1-r) \leqslant |\theta| \leqslant (1-r)\log\{(1-r)^{-1}\} \end{cases}$$

$$\log\frac{1}{1-z} = \{1+o(1)\}\log\frac{1}{1-r},$$

and so, since $z/r = 1+o(1)$,

(4) $$f(z) = \{1+o(1)\}f^*(z).$$

On the other hand, on C_2

$$\left|\frac{1}{z}\log\frac{1}{1-z}\right| \leqslant \frac{\pi}{r}+\left|\frac{1}{r}\log\left|\frac{1}{1-z}\right|\right| \leqslant \frac{\pi}{r}+\frac{1}{r}\log\frac{1}{1-r} < A\log\frac{1}{1-r},$$

(5) $$|f|+|f^*| < B\left(\log\frac{1}{1-r}\right)^{|\beta|}|1-z|^{-\alpha}.$$

We now have, since $r^{-n} = e < A$,

$$\left|a_n - \frac{1}{2\pi i}\int_C\frac{f^*(z)dz}{z^{n+1}}\right| = \frac{1}{2\pi}\left|\int_C\frac{(f-f)^*dz}{z^{n+1}}\right|$$

$$< A\int_{C_1}|f-f^*|\,d\theta + A\int_{C_2}(|f|+|f^*|)\,d\theta.$$

(6) $$< o\left(\int_{C_1}|f^*|\,d\theta\right)+B\left(\log\frac{1}{1-r}\right)^{|\beta|}\int_{C_2}\frac{d\theta}{|1-z|^{\alpha}}.$$

Now on C_1 we have, by (3),

$$\left|\frac{1}{1-z}\right| = \left|\frac{1}{1-r+i\theta}\right|+o(1) \sim \{(1-r)^2+\theta^2\}^{-\frac{1}{2}}$$

$$\int_{C_1}|1-z|^{-\alpha}\,d\theta \sim \int_{-\eta}^{\eta}\{(1-r)^2+\theta^2\}^{-\frac{1}{2}\alpha}\,d\theta = 2\int_0^{\eta}$$

$$= 2(1-r)^{1-\alpha}\int_0^{\log\{1/(1-r)\}}(1+x^2)^{-\frac{1}{2}\alpha}\,dx$$

(7) $$\sim 2(1-r)^{1-\alpha}\int_0^{\infty} = B(1-r)^{1-\alpha},$$

since $\alpha > 1$.

On C_2

$$|1-z| \geqslant \begin{Bmatrix} r|\sin\theta| > A|\theta|, & |\theta| \leqslant \frac{1}{2}\pi \\ 1, & |\theta| > \frac{1}{2}\pi \end{Bmatrix} \geqslant A|\theta| \quad (|\theta| \leqslant \pi),$$

(8) $$\int_{C_2} \frac{d\theta}{|1-z|^{\alpha}} \leqslant B\int_{\eta}^{\pi} \theta^{-\alpha}\, d\theta < B\eta^{1-\alpha} = O\left[(1-r)^{1-\alpha}\left(\log\frac{1}{1-r}\right)^{-|\beta|}\right],$$

since $\alpha - 1 > |\beta|$.

From (5), (6), (7), and (8) we conclude that

$$\left| a_n - \frac{1}{2\pi i}\int_C \frac{f^*(z)\, dz}{z^{n+1}} \right| = o\left[(1-r)^{1-\alpha}\left(\log\frac{1}{1-r}\right)^{\beta}\right] = o\,[n^{\alpha-1}(\log n)^{\beta}].$$

Since $$\frac{1}{2\pi i}\int_C = \left(\frac{1}{r}\log\frac{1}{1-r}\right)^{\beta} \times [\textit{coefficient of } z^n \textit{ in } (1-z)^{-\alpha}]$$

$$\sim (\log n)^{\beta}\frac{\Gamma(n+\alpha)}{\Gamma(n+1)\,\Gamma(\alpha)} \sim Bn^{\alpha-1}(\log n)^{\beta}$$

we obtain finally the desired result (1).

The means $M_{\lambda}(\rho)$ may be approximated for by rather similar arguments. We can, however, deduce their behaviour from that of a_n. In fact

$$J = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| (1-z)^{-\alpha}\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{\beta} \right| d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\Sigma c_n z^n|^2\, d\theta = \Sigma\, |c_n|^2 \rho^{2n}$$

where $$\Sigma c_n z^n = (1-z)^{-\frac{1}{2}\alpha}\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{\frac{1}{2}\beta}.$$

If $\alpha < 1$, $\Sigma\,|c_n|^2$ is convergent and $J < B$ for all ρ. If $\alpha = 1$ and $\beta \neq -1$,

$$J = B\Sigma\{1+o(1)\}\, n^{-1}(\log n)^{\beta}\rho^{2n} \begin{cases} < B \quad (\beta < -1) \\ \sim B\left(\log\frac{1}{1-\rho}\right)^{\beta+1} \quad (\beta > -1)\dagger. \end{cases}$$

If $\alpha > 1$,

(9) $$J = B\Sigma\{1+o(1)\}\, n^{\alpha-2}(\log n)^{\beta}\rho^{2n}$$

$$\sim B\int_0^{\infty} u^{\alpha-2}(\log u)^{\beta} e^{-2ut}\, du = Bt^{1-\alpha}\int_0^{\infty} v^{\alpha-2}\left(\log v + \log\frac{1}{t}\right)^{\beta} e^{-2v}\, dv,$$

---

† This is proved by a comparison with the integral

$$\int^{\infty} u^{-1}(\log u)^{\beta}\rho^{2u}\, du.$$

See (9) below.

where $t = \log\frac{1}{\rho} \quad 1-\rho$. But

$$\int_0^\infty = \int_0^{\log^2 1/t} v^{a-2}\left(\{1+o(1)\}\log\frac{1}{t}\right)^\beta e^{-2v}dv + \int_{\log^2 1/t}^{2/t} v^{a-2}O\left(\log\frac{1}{t}\right)^{|\beta|} e^{-2v}dv$$
$$+\int_{2/t}^{\infty} O(v^B)e^{-2v}dv.$$

The first term on the right-hand side is

$$\{1+o(1)\}\left(\log\frac{1}{t}\right)^\beta \int_0^\infty v^{a-2}e^{-2v}dv \sim B\left(\log\frac{1}{t}\right)^\beta,$$

and the remaining terms are small compared with this, so that finally

$$J \sim B(1-\rho)^{1-a}\left(\log\frac{1}{1-\rho}\right)^\beta.$$

The integrand in $M_\lambda(\rho)$ has indices λa, $\lambda\beta$. We may therefore sum up as follows:

(10) $\quad M_\lambda(\rho) < A(a, \beta, \lambda)$ for $\lambda a < 1$ or $\lambda a = 1$, $\lambda\beta < -1$,

(11) $\quad M_\lambda(\rho) \sim A(a, \beta, \lambda)\left(\log\frac{1}{1-\rho}\right)^{\beta+1/\lambda} \quad (\lambda a = 1,\ \lambda\beta > -1)$,

(12) $\quad M_\lambda^\lambda(\rho) \sim A(a, \beta, \lambda)(1-\rho)^{-a+1/\lambda}\left(\log\frac{1}{1-\rho}\right)^\beta (\lambda a > 1)$.

The exceptional cases omitted in this section require further analysis (and lead to repeated logarithms). They have little practical importance, and it is not worth while to pursue the subject further.

8.5. We consider next "Weierstrass's non-differentiable function"

$$f(z) = \Sigma a^{cn} z^{a^n},$$

where a is an integer greater than 1, and c a real constant. Here also we shall not elaborate results beyond the requirements of our applications.

If $c < 0$, $f(z)$ is continuous in γ.

For $c > 0$, we prove the following results:

(1) $$|f| < A(a, c)(1-\rho)^{-c}.$$

(2) *Given* $c > 0$, *then if* $\log a > \mathrm{Max}(2, 3c^{-1})$ *there exists a sequence*

$\rho_1, \rho_2, \ldots$ *tending to* 1, *and such that* $|f| > A(a, c)(1-\rho)^{-c}$ *for* $\rho = \rho_n$ (*and all* θ).

For (1) we have, writing B for $A(a, c)$, $b = \log a$,

$$s_n = a_0 + a_1 + \ldots + a_n = \sum_{a^m \leqslant n} a^{cm} < B \operatorname{Max} a^{cm} < B(n+1)^c$$

$$|f(z)| \leqslant \Sigma a_n \rho^n = (1-\rho)\, \Sigma s_n \rho^n < (1-\rho)\, B \Sigma (n+1)^c \rho^n < B(1-\rho)^{-c}.$$

Consider now (2); we take $\rho_n = \exp(-ca^{-n})$ and show that for $\rho = \rho_n$ the series for f is dominated by a single term.

If $u_m = a^{cm} z^{a^m}$, we have

$$\left|\frac{u_{n-s}}{u_n}\right| \leqslant e^{-cbs} \rho^{-a^n} = e^{c-cbs} \quad (s = 1, 2, \ldots, n),$$

$$\left|\frac{u_{n+s}}{u_n}\right| = e^{cbs} \rho^{a^n(a^s-1)} = \exp\{cbs - c(e^{bs}-1)\} < \exp(-\tfrac{1}{2}cb^2s^2) \leqslant \exp(-\tfrac{1}{2}cbs)$$

$(s > 0)$.

Hence

$$\left|\frac{f}{u_n}\right| \geqslant 1 - \sum_{s=1}^{n} \left|\frac{u_{n-s}}{u_n}\right| - \sum_{s=1}^{\infty} \left|\frac{u_{n+s}}{u_n}\right| > 1 - \frac{e^{c-cb}}{1-e^{-cb}} - \frac{e^{-\frac{1}{2}cb}}{1-e^{-\frac{1}{2}cb}}$$

$$> 1 - \frac{e^{-\frac{1}{2}cb}}{1-e^{-cb}} - \frac{e^{-\frac{1}{2}cb}}{1-e^{-\frac{1}{2}cb}} = \frac{1-2e^{-\frac{1}{2}cb}-2e^{-cb}}{1-e^{-cb}} > \frac{1-2e^{-\frac{3}{2}}-2e^{-3}}{1-e^{-cb}}$$

$$= \frac{A}{1-e^{-cb}} > A.$$

Also $$|u_n| = Ba^{cn} = B\left(\log\frac{1}{\rho}\right)^c \sim B\left(\frac{1}{1-\rho}\right)^c.$$

8.6. To prove our next results we require two lemmas.

LEMMA γ. *For* $s = \sigma + ti$, $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $t > 1$ *we have*

$$|\Gamma(-s)| < Bt^B e^{-\frac{1}{2}\pi t}, \quad B = A(\sigma_1, \sigma_2).$$

This follows easily from the well known asymptotic formula for $\log \Gamma(s)$.

LEMMA δ ("Mellin's integral"). *For* $\mathfrak{R}y > 0$, $\kappa > 0$ *we have*

$$e^{-y} = \frac{1}{2\pi i}\int_{-\kappa-i\infty}^{-\kappa+i\infty} \Gamma(-s)\, y^s\, ds.$$

We have, with the contours marked in the figure,

$$\frac{1}{2\pi i}\int_{(1)+(2)+(3)+(4)}\Gamma(-s)\,y^s\,ds=\sum_{m=0}^{n}\frac{(-y)^m}{m!},$$

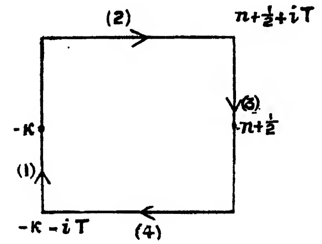

Fig. 1.

the right-hand being minus the sum of the residues at $s=0, 1, \ldots, n$. We now fix n and make $T\to\infty$. It follows at once from Lemma γ that $\int_{(1)}$ tends to $\int_{-\kappa-i\infty}^{-\kappa+i\infty}$ and $\int_{(2)}$, $\int_{(3)}$ tend to 0. Thus

$$\frac{1}{2\pi i}\int_{-\kappa-i\infty}^{-\kappa+i\infty}-\sum_{0}^{n}\frac{(-y)^m}{m!}=\frac{1}{2\pi i}\int_{n+\frac{1}{2}-i\infty}^{n+\frac{1}{2}+i\infty}\Gamma(-s)\,y^s\,ds=R_n.$$

Now

$$|\Gamma(-n-\tfrac{1}{2}-ti)|=\left|\frac{\Gamma(\frac{1}{2}-ti)}{\prod\limits_{m=0}^{n}(-m-\frac{1}{2}-ti)}\right|\leqslant\frac{2\,|\Gamma(\frac{1}{2}-ti)|}{n!}$$

$$=\frac{2}{n!}\,|\Gamma(\tfrac{1}{2}-ti)\,\Gamma(\tfrac{1}{2}+ti)|^{\frac{1}{2}}$$

(1) $$=\frac{2}{n!}\left(\frac{\pi}{\cosh\pi|t|}\right)^{\frac{1}{2}}<\frac{A}{n!}\,e^{-\frac{1}{2}\pi|t|}.$$

Hence, writing $y=\eta e^{\phi i}$, where $|\phi|<\frac{1}{2}\pi$, we see that the integrand in R_n has a modulus less than

$$\frac{A\eta^{n+\frac{1}{2}}}{n!}\,e^{-(\frac{1}{2}\pi-|\phi|)|t|}.$$

It follows that $R_n\to 0$, and this proves the lemma.

8.7. Weierstrass's function satisfies the following identity.

If $b = \log a > 0$ *(but* a *is not necessarily integral),* $\mathfrak{R}y > 0$, c *is real and not zero or a negative integer, then*

$$\text{(1)} \quad \Sigma a^{cn} e^{-ya^n} = \frac{1}{b} \sum_{n=-\infty}^{\infty} \Gamma\left(c + \frac{2\pi i n}{b}\right) y^{-c-(2\pi i n/b)} - \sum_{n=0}^{\infty} \frac{(-y)^n}{n!\,(a^{c+n}-1)}.$$

Let $y = \eta e^{\phi i}$, $|\phi| < \frac{1}{2}\pi$. Consider now

$$I(M, N) = \frac{1}{2\pi i} \int \frac{\Gamma(-s)\, y^s}{1 - a^{c+s}}\, ds$$

taken round the contour of the figure, s_r being the point $-c + 2r\pi i/b$. In this we first make $M \to \infty$. On the horizontal boundaries

$$|t| = h = (2M+1)\pi/b, \quad |y|^s \leqslant \eta^\sigma e^{|\phi t|} < K e^{|\phi t|},$$

$$|\Gamma(-s)| < K h^K e^{-\frac{1}{2}\pi h} < K e^{-KM},$$

where K's are independent of M and σ $[K = A(b, c, \kappa, \eta, \phi, N)]$;

$$a^{c+s} = \exp\{b(c+\sigma) \pm b(2N+1)\pi i/b\} \quad \text{is real and negative.}$$

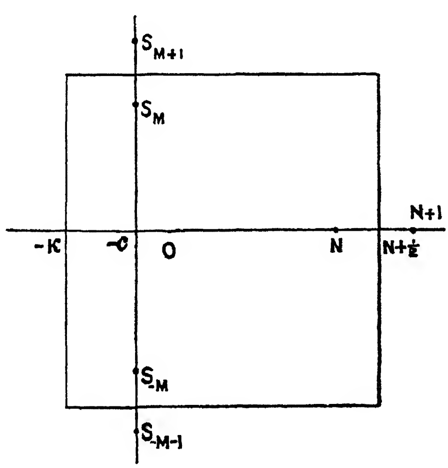

Fig. 2.

Hence the integrand has a modulus less than Ke^{-KM} and the integrals along the horizontal boundaries tend to 0. Hence, also, the series of residues at the poles on $\sigma = -c$ converges at each end.

On the vertical boundaries $|1/(1-a^{c+s})| < K$ and the integrals converge absolutely (when taken to ∞) with the integral of Lemma δ.

We have, therefore, taking account of the residues in the rectangle,

$$\frac{1}{2\pi i}\left(\int_{-\kappa-i\infty}^{-\kappa+i\infty}-\int_{N+\frac{1}{2}-i\infty}^{N+\frac{1}{2}+i\infty}\right)\frac{\Gamma(-s)\,y^s\,ds}{1-a^{c+s}}$$

$$=\sum_0^N\frac{(-y)^n}{n!(1-a^{c+n})}+\frac{1}{b}\sum_{-\infty}^{\infty}\Gamma\left(c+\frac{2\pi in}{b}\right)y^{-c-(2\pi in/b)}.$$

In this we now make $N\to\infty$. Since $1/(1-a^{c+s})<1$ for $\sigma=N+\frac{1}{2}>A(c)$, the integral along $\sigma=N+\frac{1}{2}$ tends to 0 in virtue of (1) of § 8.6, and we have

(2) $$J=\frac{1}{2\pi i}\int_{-\kappa-i\infty}^{-\kappa+i\infty}\frac{\Gamma(-s)\,y^s\,ds}{1-a^{c+s}}$$

$$=\sum_0^\infty\frac{(-y)^n}{n!(1-a^{c+n})}+\frac{1}{b}\sum_{-\infty}^{\infty}\Gamma\left(c+\frac{2\pi in}{b}\right)y^{-c-(2\pi in/b)}.$$

Now $$J=\frac{1}{2\pi i}\int_{-\kappa-i\infty}^{-\kappa+i\infty}\sum_{n=0}^{\infty}a^{cn}(ya^n)^s\Gamma(-s)\,ds,$$

and in this we may invert the order of the operations $\int$ and Σ, since

$$\int\Sigma\,|a^{cn}(ya^n)^s\Gamma(-s)|\,|ds|=\int_{-\infty}^{\infty}\Sigma a^{-(\kappa-c)n}\,\eta^{-K}\,e^{-\phi t}\,|\Gamma(-s)|\,dt$$

exists. But this leads at once, by Lemma δ, to

(3) $$J=\Sigma a^{cn}e^{-ya^n},$$

and from (2) and (3) we obtain the desired result (1).

8.8. We can now discuss our last special function. We prove:

Given $\beta>0$ and a real $c\neq 0$ satisfying $|c|<\frac{1}{4}$†, then

$$f(z)=\Sigma a_n z^n=\Sigma n^{c-\frac{1}{2}}e^{i\beta n\log n}z^n$$

satisfies

(1) $$f(z)=e^{\frac{1}{4}\pi i}\frac{b^{c-\frac{1}{2}}}{(2\pi)^c}\,t^c F(y)+\psi,$$

for $z=\rho e^{\theta i}$, $|\theta|\leqslant\pi$, where

$$\log a=b=2\pi\beta^{-1},\quad y=\sigma+ti,\quad t=\beta\exp(-1-\theta\beta^{-1}),$$

$$\sigma=\beta^{-1}t\,\log(1/\rho),$$

† The assumption $|c|<\frac{1}{4}$ enables us to avoid some minor complications. The important values of c are small positive and small negative ones.

ψ is a continuous function of ρ, θ, in $\rho \leqslant 1$, $|\theta| \leqslant \pi$, and

$$F(y) = \Sigma a^{cn} e^{-ya^n}.$$

Evidently t lies between two constants of the form $A(\beta)$, and $\sigma \to 0$ uniformly in θ as $\rho \to 1$. We suppose first $\frac{1}{2} < \rho < 1$, so that $\sigma < A(\beta)$. Now we have as $n \to \infty$, the constants of O's being independent of θ (or t), σ, n,

$$\Gamma\left(c+\frac{2\pi in}{b}\right) = (2\pi)^{\frac{1}{2}}\left(c+\frac{2\pi in}{b}\right)^{-\frac{1}{2}} \exp\left\{\left(c+\frac{2\pi in}{b}\right)\log\left(c+\frac{2\pi in}{b}\right) - \left(c+\frac{2\pi in}{b}\right) + O\left(\frac{1}{n}\right)\right\}$$

$$= e^{-\frac{1}{4}\pi i} b^{\frac{1}{2}} n^{-\frac{1}{2}}$$

$$\times \exp\left\{\left(c+\frac{2\pi in}{b}\right)\left[\log\frac{2\pi}{b} + \log n + \tfrac{1}{2}\pi i + \frac{bc}{2\pi in} + O\left(\frac{1}{n^2}\right) - 1\right] + O\left(\frac{1}{n}\right)\right\}$$

(2) $$= e^{-\frac{1}{4}\pi i} b^{\frac{1}{2}} n^{-\frac{1}{2}}$$

$$\times \exp\left\{\frac{2\pi}{b} in \log n - \frac{\pi^2}{b} n + \frac{2\pi}{b}\left(\log\frac{2\pi}{b} - 1\right) in + c\log\frac{2\pi n}{b} + \tfrac{1}{2}\pi ci + O\left(\frac{1}{n}\right)\right\}$$

$$y^{-c-(2\pi in/b)} = \exp\left\{-\left(c+\frac{2\pi in}{b}\right)\left[\log t + \tfrac{1}{2}\pi i + \log\left(1+\frac{\sigma}{ti}\right)\right]\right\}$$

$$= \exp\left\{-\left(c+\frac{2\pi in}{b}\right)\left[\log t + \tfrac{1}{2}\pi i - \frac{i\sigma}{t} + O(\sigma^2)\right]\right\}$$

(3) $$= \exp\left\{\frac{\pi^2 n}{b} - \frac{2\pi\sigma}{tb} n - \frac{2\pi}{b} in \log t - c\log t - \tfrac{1}{2}\pi ci + O(\sigma) + O(\sigma^2 n)\right\}.$$

From (2) and (3) we obtain, after a little rearrangement,

$$\Gamma\left(c+\frac{2\pi in}{b}\right) y^{-c-(2\pi in/b)}$$

$$= e^{-\frac{1}{4}\pi i}(2\pi)^c b^{\frac{1}{2}-c} t^{-c} n^{c-\frac{1}{2}} e^{\beta in \log n} (\rho e^{\theta i})^n \exp\left\{O\left(\frac{1}{n}\right) + O(\sigma) + O(\sigma^2 n)\right\}.$$

(4) $$= e^{-\frac{1}{4}\pi i}(2\pi)^c b^{\frac{1}{2}-c} t^{-c} a_n z^n + n^{c-\frac{1}{2}} v_n \rho^n,$$

where

$$|v_n| \leqslant K\left|\exp\left\{O\left(\frac{1}{n}\right) + O(\sigma) + O(\sigma^2 n)\right\} - 1\right| = K|\exp w_n - 1|.$$

We show now that

(5) $$n^{c-\frac{1}{2}} |v_n| \rho^n < K n^{-\frac{3}{2}\frac{1}{6}} \quad (\textit{uniformly in } \rho < 1).$$

We have $\sigma < K\log(1/\rho)$. Hence for $n \leqslant \nu = (\log 1/\rho)^{-\frac{9}{8}}$ we have

$$\sigma^2 n < K(\log 1/\rho)^{\frac{7}{8}}, \quad |w_n| < K, \quad |v_n| \leqslant K|w_n|,$$

and so
$$|v_n| < Kn^{-1} + K\nu^{-\frac{7}{9}} < Kn^{-\frac{7}{9}};$$

and (5) follows since $c \leqslant \frac{1}{4}$. If, on the other hand, $n > \nu$ and $1-\rho > K$, then $\nu > K$, and $\rho = \exp(-\nu^{-\frac{8}{9}})$ gives

$$n^{c-\frac{1}{2}}|v_n|\rho^n < \exp(-n\nu^{-\frac{8}{9}})\exp(K + K\sigma^2 n) < \exp(-n\nu^{-\frac{8}{9}} + K + K\nu^{-\frac{16}{9}}n)$$

$$< \exp(-Kn\nu^{-\frac{8}{9}} + K)$$

$$< \exp(-Kn\nu^{-\frac{8}{9}}) < Kn^{-\frac{37}{36}}$$

Thus (5) is proved for $1-K < \rho < 1$. The rest is easy. The series $\Sigma n^{c-\frac{1}{2}} v_n \rho^n$ is majorized by $K\Sigma n^{-\frac{37}{36}}\rho^n$ and represents a continuous function; and since $\sigma < K$, and so $|y| < K$, the series

$$S = \Sigma \frac{(-y)^n}{n!\,(a^{c+n}-1)}$$

is also continuous. Hence, since t^c is continuous and lies between two constants $A(\beta)$, it follows from the formula of § 8 . 7 that (1) holds subject to the condition $\rho > 1-K$. But when $\rho \leqslant 1-K$, and so $\sigma > K$, the functions $f(z)$, $F(y)$ are clearly continuous in (ρ, θ). The result (1) holds, therefore, without restriction.

If now we select a as in § 8 . 7 we deduce the following results concerning the behaviour of $f(z) = \Sigma n^{c-\frac{1}{2}} e^{i\beta n \log n} z^n$:

Given a real $c \neq 0$ subject to $|c| < \frac{1}{4}$ there exists a $\beta = A(c)$ with the following properties:

(*a*) *If $c > 0$ then*

$$|f(z)| < A(c)(1-\rho)^{-c} \quad (\rho < 1)$$

on the one hand, and on the other there exists a sequence (ρ_n) tending to unity and such that

$$|f(z)| > A(c)(1-\rho)^{-c}$$

for $\rho = \rho_n$ and all θ. In this case all $M_\lambda(\rho, f)$ are unbounded, and, indeed, of the same order; and this can occur with a function f for which $|a_n| = n^{-\frac{1}{2}+\delta}$.

(*b*) *If $c < 0$ then $f(z)$ is continuous in $\rho \leqslant 1$. This can occur for a function with $|a_n| = n^{-\frac{1}{2}-\delta}$.*

CHAPTER I.

The contents of this Chapter are almost all classical. It ends with the theory of the conformal representation of "schlicht" domains in general; but much of the earlier part has great intrinsic importance.

9. *The maximum modulus principle.*

9.1. THEOREM 101.†—*Let $f(z)$ be regular in a bounded domain D. For each point ξ of the boundary suppose that, for some $\delta = \delta(\epsilon, \xi)$,*

$$|f| < M+\epsilon$$

for all z of D in $|z-\xi| < \delta$. [We call these "conditions (A)".] *Then $|f| \leqslant M$ in D, and equality does not occur unless f is a constant.*

[N.B.—f is defined only in D, *i.e.* at *interior* points.] The proof is very much like that of Theorem 49. Let G, possibly ∞, be the upper bound of $|f|$ in D. Consider the class of points P such that every neighbourhood S of P gives G as upper bound of $|f|$ in SD. The bisection argument shows that there exists at least one P in D' (not necessarily in D). Suppose now no internal point is a P; then some boundary point ξ is, and since $|f| < M+\epsilon$ in *some* neighbourhood of ξ, we must have $G \leqslant M$. Since evidently $|f| < G$ at all interior points [equality makes z a P], we have also $|f| < M$.

I prove now that if an internal point is a P, then $f = C$, a constant in D, and $|C| = G$. This will complete the proof, since every ξ will be a P, and so $G \leqslant M$ as before.

Suppose an interior point z_0 is a P. By continuity $|f(z_0)| = G$, say $f(z_0) = Ge^{i\alpha}$. Then

$$f(z_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(z_0+re^{i\theta})\,d\theta,$$

where $|z-z_0| \leqslant r$ is any circle round z_0 lying entirely in D. Thus

$$G = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\alpha} f\,d\theta.$$

† The theorems of Chapter I begin at number 101, those of Chapter II at 201, and so on. The sections are numbered consecutively.

Since $|e^{-i\alpha}f| \leqslant G$ and $|f|$ is continuous we must have $|f| = G$ on the whole circle. If now $f(z_0+re^{i\theta}) = Ge^{i\phi}$,

$$G = \frac{1}{2\pi}\int_0^{2\pi} Ge^{i(\phi-\alpha)}\,d\theta = \frac{G}{2\pi}\int_0^{2\pi}\cos(\phi-\alpha)\,d\theta,$$

and the same argument shows that $\cos(\phi-\alpha) = 1$ on the whole circle, and so $f = Ge^{i\alpha}$. Since r is arbitrary, we have $f = C = Ge^{i\alpha} = f(z_0)$ in any circle round z_0 lying in D.

Suppose now f is not equal to C everywhere in D, say $f \neq C$ for $z = z_1$. Join z_0, z_1 by a polygon, and let ζ be determined, by a Dedekind section, so that $f = C$ on the polygon from z_0 to anything short of ζ, but not from z_0 to anything beyond ζ. By continuity $f(\zeta) = C$. Hence by the above argument $f = C$ in a circle round ζ, and this is false.

An important particular case of conditions (A) occurs when f is continuous in D' and $|f| \leqslant M$ at all boundary points.

COROLLARY 1.—*The result of the theorem is true also if f is regular at each point of D and $|f|$ is one-valued in D.*

COROLLARY 2.—*If $f \neq 0$ in D there is a minimum modulus principle.* We have only to consider $g = 1/f$.

COROLLARY 3.—*If f is regular and never zero in D, f is continuous in D', and the boundary values of $|f|$ are everywhere constant, then f is constant in D.*

By Corollary 2 $|f|$ is not less than its boundary value anywhere in D; hence $|f|$ attains its upper bound at an interior point and f is a constant.

Corollary 3 becomes false if the condition $f \neq 0$ is omitted.

9.2. *The rôle of Cauchy's theorem in the above proof.*

1. The theorem is used only in the case of a circular contour within the circle of convergence.

2. Its use is avoidable. It is enough to show that $|f|$ cannot be a *maximum* at an interior point of D, unless f is a constant. If f is not constant, and the point is the origin, we have

$$f = a_0 + c_n z^n + \ldots, \quad c_n \neq 0,$$

Let δ be small and positive and let z be a root of the equation

(1) $$z^n = \delta a_0/c_n.$$

Then $$|f| = (1+\delta)|a_0| + O(\delta^2) > |a_0| = |f(0)|.$$

It is instructive to compare the problem of proving the fundamental theorem of algebra without using Cauchy's theorem (or an equivalent). This problem, too, reduces, on the above lines, to the existence of a z satisfying (1), and is substantially equivalent to the existence of a solution of the equation $z^n = k$. This last existence theorem can be proved, if with some difficulty, by purely elementary reasoning.

9.3. The following argument has a certain interest in spite of requiring assumptions more stringent than (A). Suppose D bounded by a contour for which Cauchy's theorem is valid, f continuous in D' (and regular in D), and $|f| \leqslant M$ on the boundary.

(i) Let z_0 be interior to D. Then

$$|f(z_0)|^n = \left| \frac{1}{2\pi i} \int_C \frac{f^n(z)\,dz}{z-z_0} \right| \leqslant \frac{L}{2\pi\delta} M^n,$$

where $L =$ length of C, $\delta =$ distance of z_0 from C.

Make $n \to \infty$: $$|f(z_0)| \leqslant M.$$

(ii) Suppose $|f(z_0)| = M$: to prove f constant. We have

$$|nf^{n-1}(z_0).f'(z_0)| = \left| \frac{1}{2\pi i} \int_C \frac{f^n\,dz}{(z-z_0)^2} \right| \leqslant \frac{L}{2\pi\delta^2} M^n.$$

Hence $$|f'(z_0)| \leqslant \frac{L}{2\pi\delta^2} \cdot \frac{M}{n},$$ and $n \to \infty$ gives $f'(z_0) = 0$.

Let $F = f^n$. Then $F'' = n(n-1)f^{n-2}(f')^2 + nf^{n-1}f''$, and so

$$F''(z_0) = n\{f(z_0)\}^{n-1} f''(z_0)$$

$$|nM^{n-1}f''(z_0)| = |F''(z_0)| = \left| \frac{2!}{2\pi i} \int_C \frac{f^n\,dz}{(z-z_0)^3} \right| \leqslant \frac{2!\,L}{2\pi\delta^3} M^n$$

$$f''(z_0) = 0.$$

Similarly $f'''(z_0) = 0$, etc., and f is a constant.

9.4. THEOREM 102.—*Let $f(z)$ be regular in a bounded D; $|f| < M + \epsilon$ for some neighbourhood of each boundary point, except for a finite set $\xi_1, \ldots, \xi_n$; $|f| < M'$ in D.* [Conditions (B).] *Then $|f| \leqslant M$ in D, and $|f| < M$ unless f is a constant.*

Let d be the diameter of D; $\sigma > 0$. Let

$$\phi = \omega^\sigma f, \quad \omega = \Pi\left(\frac{z-\xi_r}{d}\right).$$

ω^σ is regular *at every point of* D [*in* D if D is simply connected], and $|\omega^\sigma|$ is one-valued in D. The same things are therefore true of ϕ. Also $|\omega^\sigma| \leqslant 1$, so $|\phi| \leqslant |f|$, and $|\phi| < M+\epsilon$ in some neighbourhood of any ξ other than a ξ_r. Also

$$|\phi| \leqslant |f| d^{-\sigma} |z-\xi_r|^\sigma \leqslant M' d^{-\sigma} \delta^\sigma < \epsilon \leqslant M+\epsilon$$

if $|z-\xi_r| < \delta$ and $\delta = \delta(\epsilon, \sigma)$ is suitably chosen. Hence $|\phi| < M+\epsilon$ in some neighbourhood of each ξ_r, and ϕ satisfies the conditions of Theorem 101, Corollary 1. Hence

$$|\phi| \leqslant M \text{ in } D,$$

$$|f| \leqslant M\left(\frac{d^n}{\Pi|z-\xi_r|}\right)^\sigma \text{ in } D.$$

Fixing z and making $\sigma \to 0$ we have $|f| \leqslant M$. By Theorem 101 f is constant if $|f| = M$ at an interior point.

9.5. *Strip-theorems.*

9.51. The principle of the argument of Theorem 102 generalizes as follows:

THEOREM 102*a*.—*Suppose that f is regular in D, and satisfies $|f| < M+\epsilon$ for some neighbourhood of each ξ of the boundary, except for ξ's of a set E; and that a function $\omega(z)$ exists, regular at every point of D, and not identically zero, for which $|\omega(z)|$ is one-valued in D and satisfies $|\omega| \leqslant 1$ in D. Suppose finally that for any given positive ϵ and σ the inequality $|\omega^\sigma f| < M+\epsilon$ is satisfied in some neighbourhood of each ξ of E. Then $|f| \leqslant M$ in D.*

$g = \omega^\sigma f$ satisfies the conditions of Theorem 101, Cor. 1. If z_0 is an interior point, not a zero of ω,

$$|g(z_0)| \leqslant M, \quad |f(z_0)| \leqslant M|\omega(z_0)|^{-\sigma}$$

and $|f(z_0)| \leqslant M$ by making $\sigma \to 0$. If z_0 is a zero of ω it has points z_1, not zeros of ω, arbitrarily near it; $|f(z_1)| \leqslant M$, and so $|f(z_0)| \leqslant M$ by continuity.

9.52. Theorem 102a has applications to the theory of functions regular in infinite vertical strips. We nowhere happen to use these applications, but they have great intrinsic interest, and we shall digress to discuss them.

THEOREM 103.—*Suppose*

(i) f *is regular in a half-strip* $\alpha < x < \beta$, $y > \eta$, *or* D;

(ii) $|f| \leqslant M+\epsilon$ *in some neighbourhood of every (finite) point of the boundary;*

(iii) $f = O(\exp\{e^{\vartheta\pi|z|/(\beta-\alpha)}\})$ *uniformly in* D, *where* $\vartheta < 1$.

Then $$|f| \leqslant M \text{ in } D.$$

Remarks. (1) The strip is the transform of a bounded D with one exceptional ξ_1, ξ_1 going to ∞.

(2) As against Theorem 102 we assume a highly specialized boundary, with a cusp at ∞, but, on the other hand, much less than $|f| = O(1)$. There are other compromise theorems.

(3) The theorem is more or less best possible. The example $\alpha = -\frac{1}{2}\pi$, $\beta = \frac{1}{2}\pi$, $\eta = 0$, $f = \exp(e^{-iz})$, shows that $\vartheta = 1$ is not permissible; here $|f| = 1$ on $x = \pm\frac{1}{2}\pi$, $|f| = \exp(e^y)$ on $x = 0$.

Proof.—As often happens, the critical case gives a clue. Let α, β, η be as above, $\sigma > 0$, $g = f\exp(-\sigma e^{-ikz}) = f\omega^\sigma$, where $\vartheta < k < 1$. Clearly $|\omega| \leqslant 1$ in D. As $y \to \infty$

$$g = O(\exp[e^{\vartheta|z|} - \sigma e^{ky}\cos\tfrac{1}{2}k\pi]) \qquad \text{(uniformly in } x).$$

Now $e^{\vartheta|z|} - \sigma e^{ky}\cos\frac{1}{2}k\pi \leqslant e^{\vartheta(y+\frac{1}{2}\pi)} - \sigma e^{ky}\cos\frac{1}{2}k\pi \to -\infty$ (uniformly in x),

since $k > \vartheta$. Hence $g \to 0$ uniformly, and so $|g| < M+\epsilon$ $(y > y')$. The theorem now follows from Theorem 102a, transformed to the case $\xi_1 = \infty$.

To carry out the details, not using Theorem 102a, we proceed as follows. Let z_0 be a point of D. We can choose H so that $|g(z)| < M+\epsilon$ on $y = H$, and may suppose $H > y_0 = \mathfrak{I}z_0$. Since $|\omega^\sigma| \leqslant 1$ we have now $|g| < M+\epsilon$ in some neighbourhood of every boundary point of

the *rectangle*, therefore $|g| \leqslant M$ at $z = z_0$ by Theorem 101. That is,

$$|f(z_0)| \leqslant M|e^{\sigma e^{-ikz_0}}|.$$

In this we make $\sigma \to 0$.

COROLLARY.—*Suppose* (i) *and* (iii) *hold, and f is continuous on the boundary; $f = O(y^a)$, $O(y^b)$ on $x = \alpha, \beta$. Then $f = O(y^c)$ on $x = \gamma$, uniformly in γ, where $c = p\gamma + q$ and $px+q$ is the linear function that is a at $x = \alpha$ and b at $x = \beta$.*

We may suppose $\eta > 0$. Let $\phi(z) = f(z)(-zi)^{-(pz+q)} = f\psi$, regular in D, continuous in D'.

$$\begin{aligned}
|\psi| &= |(y-ix)^{-(px+q)-ipy}| \\
&= |y-ix|^{-(px+q)} \exp\left(py\,\mathfrak{I}\log\frac{y-ix}{y}\right) \\
&= y^{-(px+q)}\left|1+O\left(\frac{1}{y}\right)\right|^{O(1)} \exp\left\{py\left[-\frac{x}{y}+O\left(\frac{1}{y^2}\right)\right]\right\} \\
&= y^{-(px+q)}e^{-px}\left\{1+O\left(\frac{1}{y}\right)\right\} \qquad (O\text{'s uniform}).
\end{aligned}$$

Thus $\phi = O(1)$ on $x = \alpha$, $x = \beta$, and, of course, on the bottom boundary. Since $\psi = O(y^K) = O(|z|^K)$ condition (iii) holds for ϕ if ϑ is rechosen. Hence $\phi = O(1)$ uniformly in the strip, whence the result for f.

A similar theorem holds for $O\{(\log y)^{a,\,b}\}$, etc.

9.53. THEOREM 104.—*Suppose* (i) *and* (iii) *hold, and f is continuous on the boundary. Also that $\overline{\lim}_{y\to\infty} |f| \leqslant M$ on $x = \alpha$ and $x = \beta$. Then $\overline{\lim}\,|f| \leqslant M$ uniformly in the strip.*

f is bounded on the boundary, therefore, by Theorem 103, in D'. Suppose $\eta \geqslant 0$; and let $H = H(\epsilon)$ be the ordinate beyond which $|f| < M+\epsilon$ on the edges. Let h be a positive constant, $g = fz/(z+hi)$. In any case $|z/(z+hi)| < 1$ in the strip; we suppose further $h = h(H)$ chosen so (large) that

$$|g| < M+\epsilon \quad \text{on} \quad y = H$$

(this is possible since $|g| \leqslant |f| < K$). Then g satisfies the conditions of Theorem 103 in the strip above $y = H$. Thus $|g| \leqslant M+\epsilon$ in this strip,

and $\overline{\lim}_{z\to\infty} |g| \leqslant M+\epsilon$, uniformly. This gives $\overline{\lim}|f| \leqslant M+\epsilon$† since $z/(z+hi) \to 1$ uniformly (in x). Hence $\overline{\lim}|f| \leqslant M$.

COR. 1.—*If, subject to* (i), (iii), *and the continuity of f on the boundary,*

$$\overline{\lim}|f| \leqslant \begin{cases} a & x=\alpha, \\ b & x=\beta, \end{cases}$$

where $a, b \neq 0$, then $\overline{\lim}_{y\to\infty}|f| \leqslant e^{px+q}$ (uniformly), p and q being chosen so as to make the right-hand side a at α and b at β.

Consider $$g = fe^{-(pz+q)}.$$

COR. 2.—*If $f=O(1)$ on $x=\alpha$, $f=o(1)$ on $x=\beta$, then $f=o(1)$ on $x=\gamma$ if $\alpha<\gamma\leqslant\beta$.*

We may take $b=\epsilon$ in Cor. 1, observing that $e^{p\gamma+q}$ (for fixed γ) tends to 0 with ϵ.

This ends our digression.

9.61. *The conjugate function $\bar{F}(z)$.* Suppose $F(z)$ is regular in D. We define $\bar{F}(\zeta)$, for a ζ of $\bar{D}$, to be the conjugate of the number $F(\bar{\zeta})$. *Then $\bar{F}(\zeta)$ is a regular analytic function of ζ in $\bar{D}$.* For it is evidently one-valued; also if $\zeta=\bar{z}$, $\zeta+\delta\zeta=\overline{z+\delta z}$ and so $\delta\zeta=\overline{\delta z}$,

$$\frac{\bar{F}(\zeta+\delta\zeta)-\bar{F}(\zeta)}{\delta\zeta} = \frac{\overline{F(z+\delta z)}-\overline{F(z)}}{\overline{\delta z}} \to \overline{F'(z)}$$

(by reason of the existence of the corresponding limit for conjugates), and $\bar{F}$ is differentiable at ζ.

9.62. It is convenient to take here a proposition quite unconnected with our immediate topic.

THEOREM 105.—(*Schwarz's continuation theorem, or "symmetry principle".) Suppose the domain D has a segment of a straight line AB as a free part of the boundary (i.e. if P is between A and B the whole interior of some semicircle about P lies in D), and that D lies wholly*

† *Not M.* It is $M+\epsilon$ that plays the part played in Theorem 103 by M: the *domain* and g depend on ϵ.

on one side of the complete straight line AB. Suppose that f is regular in D, is continuous (in D') at points of (AB), and takes real values on (AB). Let us define

$$F(z) = \begin{cases} f(z), \text{ for } z \text{ in } D+(AB), \\ \text{the conjugate of } f(\bar{z}), \text{ for } z \text{ in } \bar{D}. \end{cases}$$

Then F is a regular analytic function in $D_1 = D+\bar{D}+(AB)$.

$\bar{D}$ and $\bar{z}$ here denote reflections in AB. We may, however, suppose AB the real axis, so that $\bar{D}$, $\bar{z}$ have their usual meanings. D and $\bar{D}$ have no common point.

F is regular in $\bar{D}$, as above, also continuous (in D_1) on (AB). It is enough to show that F is regular in *some* circle about every P of (AB); for it is then regular *at* every point of D_1, and one-valued, by definition, in D_1. Let us then draw a circle C round P in D_1. Let

$$\phi(\zeta) = \frac{1}{2\pi i}\int_C \frac{F(z)\,dz}{z-\zeta},$$

a function regular in the interior of C. Let C_1, C_2 be the perimeters of the upper and lower semicircles (including the diameter). If now ζ does not lie on the real axis

$$2\pi i\,\phi(\zeta) = \int_{C_1}\frac{F\,dz}{z-\zeta} + \int_{C_2}\frac{F\,dz}{z-\zeta}.$$

If ζ belongs to the upper semicircle, $\int_{C_1} = 2\pi i\,F(\zeta)$, since F is regular inside C_1 and continuous on the boundary, and ζ is in C_1. Also $\int_{C_2} = 0$, since $F/(z-\zeta)$ is regular inside C_2 and continuous on the boundary. Thus $\phi(\zeta) = F(\zeta)$. Similarly this is true when ζ is in the lower semicircle. Hence ϕ and F agree in the interiors of C_1 and C_2, and so, *by the continuity of F and ϕ, also on (AB).* Thus F is identical with ϕ, a function regular in C.

9.7. THEOREM 106.—(*Another maximum modulus principle.*) *Let $f(z)$ be regular in D, $F(z)$ in $\bar{D}$. Let $\psi(z) = f(z)F(\bar{z})$ (ψ is defined in D, but is not analytic). Suppose ψ satisfies conditions (B) in D (i.e. $|\psi| < M+\epsilon$ in some neighbourhood of every ξ but a finite number, and $|\psi| < M'$). Then*

$$|f(z)F(\bar{z})| = |\psi| \leq M \text{ in } D.$$

Let $g(z) = f(z)\bar{F}(z)$, evidently regular in D and satisfying a con-

dition of type (B). We have $|\bar{F}(z)| = |F(\bar{z})|$, $|\psi| = |g| \leqslant M$, by Theorem 102.

COR.—*In the theorem $\bar{z}$ and z, $\bar{D}$ and D, may be reflections in any line.*

9.8. We propose next to explain an important *method*; this will be grasped most easily if we take a concrete example of its application; the example has no very special interest in itself. Consider the rectangle R of Fig. 3, in which we suppose b:

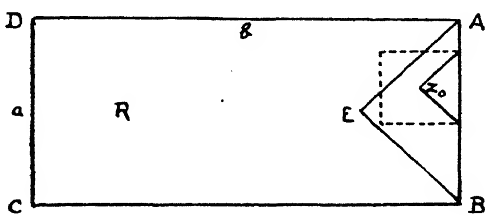

Fig. 3.

Suppose f regular in R, continuous in R'; $|f| \leqslant M$ on the whole boundary (therefore also inside), $|f| \leqslant m < M$ on AB. Then in the (isosceles right-angled) triangle AEB

$$|f| \leqslant M^{\frac{3}{4}} m^{\frac{1}{4}}.$$

Let z_0 be any point of the triangle,

$$f(z) = \phi(z - z_0) = \phi(\zeta),$$

$$\psi(\zeta) = \prod_0^3 \phi(\zeta i^n),$$

so that $\psi(\zeta)$ is a regular function of z in the dotted square. On each side of the square *one* of the ϕ's has modulus $\leqslant m$, the others moduli $\leqslant M$. Hence $|\psi| \leqslant M^3 m$, in the square. In particular,

$$|\psi(0)| \leqslant M^3 m,$$

i.e.
$$|f(z_0)|^4 \leqslant M^3 m.$$

COR.—*We may suppose conditions of type B ($M+\epsilon$ and $m+\epsilon$) instead of continuity in R'.*

For another example see Fig. 4. At any point z_0 such that a rotation π round it creates a region bounded only by m-arcs, we have

$$|f(z_0)| \leqslant \sqrt{(Mm)}.$$

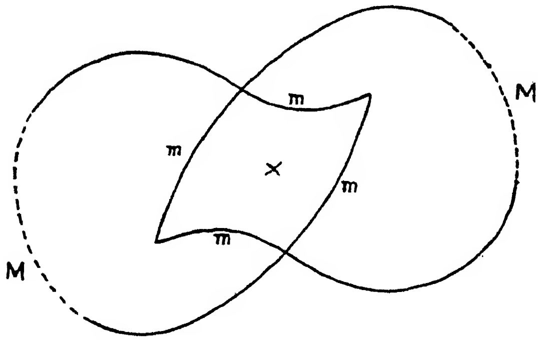

Fig. 4.

In such problems we can also often employ Theorem 106. Thus, in Fig. 5, suppose ACB an m-arc, AEB an M-arc. *If the reflection of* D_1 *in* AB *lies in the original domain, then* $|f| \leqslant \sqrt{(Mm)}$ *on* AB, *and so in* D_1.

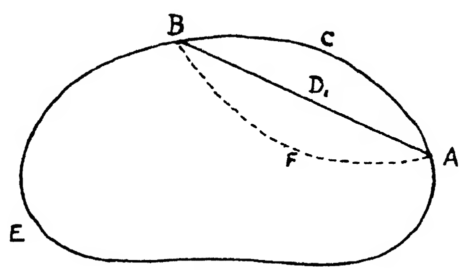

Fig. 5.

Take $F = f$ and $D = D_1 + \bar{D}_1 + (AB)$ in Theorem 106. $f(z)$ and $F(z)$ are defined in D. For a z of ACB, $|f(z)| \leqslant m$, $|f(\bar{z})| \leqslant M$ $|f(z)f(\bar{z})| \leqslant mM$. Similarly for AFB. Hence $|f(z)f(\bar{z})| \leqslant mM$ for all z of D, in particular for z of AB, where $\bar{z} = z$.

10. *Some classical theorems.*

10.1. THEOREM 107 (*"Schwarz's Lemma"*).—*Let* f *be regular, and* $|f| \leqslant M$ *in* $|z| < R$; $f(0) = 0$. *Then*

$$|f| \leqslant M|z|/R \quad (|z| < R).$$

In particular this holds if f is regular in $|z| < R$ and continuous in

We may suppose $M = 1$, $R = 1$. $\phi(z) = f/z$ is regular in the unit circle γ (by Osgood's theorem). Given $r < 1$, take ρ such that $r < \rho < 1$. On $|z| = \rho$ $|\phi(z)| \leqslant 1/\rho$. Hence $|\phi(z)| \leqslant 1/\rho$ also for $|z| = r$. Since the left side does not depend on ρ, we may make $\rho \to 1$; thus

$$|\phi| \leqslant 1, \quad |f| \leqslant |z|.$$

COR.—*Let $f(z)$ be regular, $|z| = \rho$, and $|f| \leqslant \psi(\rho)$ in γ, where ψ is increasing and $\psi(0) > 0$. Suppose also $f(0) = 0$. Then*

$$|f| \leqslant K\rho\psi(\rho) \quad (\rho < 1),$$

where K is a constant, which may be taken to be $2\psi(\frac{1}{2})/\psi(0) = K_0$.

For $\rho > \frac{1}{2}$, $\quad |f| \leqslant \rho\psi(\rho)/\rho \leqslant 2\rho\psi(\rho) \leqslant K_0\rho\psi(\rho)$.

For $\rho \leqslant \frac{1}{2}$,

$$|f/z| \leqslant \operatorname*{Max}_{|z|=\frac{1}{2}} |f/z| \leqslant 2 \operatorname*{Max}_{|z|=\frac{1}{2}} |f| \leqslant 2\psi(\tfrac{1}{2}) \leqslant 2\psi(\tfrac{1}{2})\,\psi(\rho)/\psi(0).$$

10.2. THEOREM 108 (*Hadamard's "three circles theorem"*).—*Suppose $r_1 \leqslant r_2 \leqslant r_3$; f regular and $|f| \leqslant M_3$ in $|z| \leqslant r_3$; $|f| \leqslant M_1$ in $|z| \leqslant r_1$ (or on $|z| = r_1$). Then*

$$|f| \leqslant M_1^{\vartheta} M_3^{1-\vartheta} \quad (|z| \leqslant r_2),$$

where $\quad \vartheta = \log \dfrac{r_3}{r_2} \Big/ \log \dfrac{r_3}{r_1}, \quad 1-\vartheta = \log \dfrac{r_2}{r_1} \Big/ \log \dfrac{r_3}{r_1},$

Consider $f(z)\,z^{-\lambda}$ in the annulus (r_1, r_3) and apply Theorem 101, Cor. 1, choosing λ so that $M_1 r_1^{-\lambda} = M_3 r_3^{-\lambda} = \mu$. We get, for $|z| = r_2$, $|fz^{-\lambda}| \leqslant \mu$, $M_2 \leqslant \mu r_2^{\lambda}$, and this is the desired result.

10.3. THEOREM 109.—*Suppose $f(z)$ is regular in $|z| < r$. $\mathfrak{R}f \leqslant U$ $(|z| < r)$, $f(0) = a_0 = \beta + i\gamma$. Then for $|z| = \rho < r$ we have*

$$|f(z) - a_0| \leqslant \frac{2(U-\beta)\rho}{r-\rho}, \quad |\mathfrak{R}f(z) - \beta| \leqslant \frac{2(U-\beta)\rho}{r-\rho}.$$

We have $\beta \leqslant U$. Moreover, since the desired results are true by continuity in the limiting case $U = \beta$ if they are always true for $U > \beta$, we may suppose $U - \beta > 0$. We may further suppose $a_0 = \beta + i\gamma = 0$. We suppose then $f(0) = 0$, $\mathfrak{R} f \leqslant U$, $U > 0$. We may suppose also $r = 1$.

The values of f are confined to the half-plane $\mathfrak{R}w \leqslant U$; also $f(0) = 0$.

What function $w = g(z)$ conformally represents the z-circle on the w half-plane, and $z = 0$ by $w = 0$? It is

$$g(z) = \frac{-2Uz}{1-z}.$$

Take the inverse function

$$Z(g) = \frac{g}{g-2U},$$

and consider

(1) $$\omega(z) = Z\{f(z)\} = \frac{f}{f-2U}.$$

Conformal representation theory shows that ω is regular in $|z|<1$ and $|\omega| \leqslant 1$. But without appealing to this, let us define ω by (1). Then, since $\mathfrak{R}(f-2U) \leqslant -U < 0$, ω is regular in γ, and if $f = u+iv$, we have

$$|\omega| = \sqrt{\left(\frac{u^2+v^2}{(2U-u)^2+v^2}\right)}$$

and so $|\omega| \leqslant 1$, since $2A-u \geqslant |u|$. [Consider $u > 0$, $u \leqslant 0$ separately.] But $\omega(0) = 0$. Hence, by Theorem 107,

$$|\omega(z)| \leqslant |z|,$$

$$f(z)\,[= g\{\omega(z)\}] = -\frac{2U\omega}{1-\omega},$$

$$|f| \leqslant \frac{2U|\omega|}{1-|\omega|} \leqslant \frac{2U\rho}{1-\rho}.$$

10.4. There is an alternative proof, giving more.

THEOREM 110 (*Borel*).—*Suppose* $f(z) = \Sigma a_n z^n$ *is regular in* γ *and* $\mathfrak{R}f \leqslant U$. *Then* $|a_n| \leqslant 2U_1 = 2(U-\beta)$ $(n > 0)$.

For $z = re^{i\theta}$, $r < 1$,

$$f = \beta+i\gamma+\sum_1^\infty(\beta_n+i\gamma_n)r^n(\cos n\theta+i\sin n\theta) = P(r,\,\theta)+iQ(r,\,\theta),$$

$$P = \beta+\sum_1^\infty(\beta_n\cos n\theta-\gamma_n\sin n\theta)\,r^n, \quad \text{uniformly convergent in } \theta.$$

Hence

(1) $$2\pi\beta = \int_0^{2\pi} P\,d\theta,$$

$$\pi r^n\beta_n = \int_0^{2\pi} P\cos n\theta\,d\theta, \qquad \pi r^n\gamma_n = -\int_0^{2\pi} P\sin n\theta\,d\theta,$$

(2) $$\pi r^n a_n = \int_0^{2\pi} Pe^{-ni\theta}\,d\theta.$$

(i) *Suppose* $U \geqslant 0$. *Then* $0 \leqslant |P|+P \leqslant 2U$. From (1) and (2),

$$\pi r^n |a_n| \leqslant \int_0^{2\pi} |P|\, d\theta = \int_0^{2\pi} (|P|+P)\, d\theta - 2\pi\beta$$

$$\leqslant \int_0^{2\pi} (2U)\, d\theta - 2\pi\beta = 2(2U-\beta)\,\pi.$$

Make $r \to 1$: $$|a_n| \leqslant 2(2U-\beta),$$

the result with $2U$ for U, and under an additional hypothesis.

(ii) This can be amended. With the original hypothesis let $f' = f-U$. Then $\mathfrak{R} f' \leqslant U' = 0$, $\beta' = \beta - U$. The extra condition in (i) is satisfied, and (i) gives

$$|a_n| \leqslant 2\,.\,\{2\,.\,0-(\beta-U)\} = 2(U-\beta),$$

the full result.

(iii) An alternative device. Let $f_1 = f - a_0$. Then $\mathfrak{R} f_1 \leqslant U_1$, $U_1 = U - \beta \geqslant 0$, and we have to show $|a_n| \leqslant 2U_1$. Here $f_1 = P+iQ$, where

(3) $$0 = \int_0^{2\pi} P\, d\theta,$$

$$\pi r^n a_n = \int_0^{2\pi} P e^{-ni\theta}\, d\theta.$$

Let now $\alpha = \arg a_n$; then

$$\pi r^n |a_n| = \pi r^n a_n e^{-i\alpha} = \int_0^{2\pi} P e^{-(n\theta+\alpha)i}\, d\theta = \int_0^{2\pi} P \cos(n\theta+\alpha)\, d\theta,$$

since the left side is real,

$$= \int_0^{2\pi} P\{1+\cos(n\theta+\alpha)\}\, d\theta, \quad \text{by (3)},$$

$$\leqslant \int_0^{2\pi} U_1\{1+\cos(n\theta+\alpha)\}\, d\theta = 2\pi U_1,$$

as before.

To see that Theorem 110 includes Theorem 109 we observe that

$$|f(z)| \leqslant F(\rho) = \Sigma |a_n| \rho^n \leqslant |a_0| + 2(U-\beta) \sum_1^\infty \rho^n.$$

(iv) *A function-theory proof of Theorem* 110.

Let $f_1 = \sum_1^\infty a_n z^n$, $\mathfrak{R} f_1 \leqslant U_1$. We have to show $|a_n| \leqslant 2U_1$. As in the proof of Theorem 109, we have for $|z| = \rho > 0$,

$$|f_1| \leqslant \frac{2U_1\rho}{1-\rho}.$$

Hence $$\left|\frac{f_1(z)}{z}\right| \leqslant \frac{2U_1}{1-\rho}.$$

As $z \to 0$ the left side tends to a_1. Hence

$$|a_1| \leqslant 2U_1,$$

the result for the special case $n = 1$.

Now let $\omega = e^{2\pi i/k}$; then

$$\frac{1}{k}\sum_{r=0}^{k-1} f_1(\omega^r z) = \sum_{n=1}^{\infty} a_{nk} z^{nk} = g_1(z^k) = g_1(Z).$$

The series for g_1 is convergent for $|z|<1$, and so for all Z for which $|Z|<1$; g_1 is regular in $|Z|<1$. Since $\mathfrak{R} g_1 \leqslant U_1$ we have

$$|\textit{coefficient of } Z| \leqslant 2U_1,$$

$$|a_k| \leqslant 2U_1.$$

We record finally the following deduction from Theorem 110

COR.—*Suppose that $f(z)$ is regular and*

$$\mathfrak{R}\{f(z)-f(0)\} \leqslant U_1$$

in $|z|<r$. Then

$$|f'(z)| \leqslant \frac{2U_1 r}{(r-\rho)^2} \qquad (|z| = \rho < r).$$

We may suppose $r = 1$, and the result follows from

$$|f'(z)| \leqslant \sum^{\infty} n\,|a_n|\,\rho^{n-1} \leqslant 2U_1 \Sigma n\rho^{n-1}$$

10.5. There is another result of a slightly different kind. In the first place we have

THEOREM 111.—*Suppose $f(z)$ regular in*

$$|z-z_0| \leqslant r, \quad f = P(r, \phi) + iQ(r, \phi) \quad (z = z_0 + re^{i\phi}).$$

Then $$f'(z_0) = \frac{1}{\pi r}\int_0^{2\pi} P(r, \phi) e^{-i\phi}\, d\phi.$$

We have

(1) $$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^2} = \frac{1}{2\pi r}\int_0^{2\pi} (P+iQ)\, e^{-i\phi}\, d\phi.$$

Also $$0 = \frac{1}{2\pi i r^2}\int_C f(z)\, dz = \frac{1}{2\pi r}\int_0^{2\pi} (P+iQ)\, e^{i\phi}\, d\phi.$$

In this we change the sign of i and add to (1), obtaining our result.

From Theorem 111 we deduce at once

THEOREM 112 (*Schwarz*).—*Let f be regular and $|\mathfrak{R}f| \leqslant C$ in $|z-z_0| \leqslant r$. Then*

$$|f'(z_0)| \leqslant 2C/r.$$

10.6. THEOREM 113 (*Vitali*).—*Suppose that we are given a domain D and a sequence (f_n) satisfying:*

(*a*) *f_n is regular in D for each n;*

(*b*) *f_n is uniformly bounded in every D'_-;*

(*c*) *$f_n \to$ a limit (necessarily finite) for each z of $z_1, z_2, \ldots$, an infinite sequence (of different z's) in D with at least one limit point z_0 in D. Then there exists an $f(z)$, regular in D, such that*

$$f_n \to f, \quad f_n^{(p)} \to f^{(p)},$$

uniformly in any D'_-.

It is a consequence of (*a*) and (*b*) that f_n is a continuous function of z in any D'_-, uniformly in n. In fact, let a be the distance of D'_- from $F(D)$, and let z_1, z_2 be two near points of D'_-. We have $|f_n| < K$ for all z of D distant more than $\frac{1}{2}a$ from $F(D)$ and all n. Hence, if

$$|z_1 - z_2| < \delta < \tfrac{1}{4}a$$

and C is the circle with centre z_1 and radius $\frac{1}{2}a$,

$$(1) \qquad |f_n(z_1) - f_n(z_2)| = \frac{1}{2\pi}\left|\int_C f_n(\zeta)\left(\frac{1}{\zeta - z_1} - \frac{1}{\zeta - z_2}\right) d\zeta\right|$$

$$\leqslant \frac{|z_1 - z_2|}{2\pi} \int_C \left|\frac{f_n(\zeta)\, d\zeta}{(\zeta - z_1)(\zeta - z_2)}\right|$$

$$\leqslant \frac{|z_1 - z_2|}{2\pi} \int_{-\pi}^{\pi} \frac{K \cdot \frac{1}{2}a\, d\theta}{\frac{1}{2}a \cdot \frac{1}{4}a} = \frac{4|z_1 - z_2| K}{a}$$

$$< \epsilon,$$

if $\delta < \frac{1}{4}a\epsilon/K$. Since n is arbitrary this gives the desired result.

I say now, abandoning uniformity for the moment in our conclusions, that for every z of D f_n tends to a limit f as $n \to \infty$ (through all values). If not, there exists a z^* of D at which f_n does not tend to a limit, and we can find two subsequences (n'_r), (n''_r) through which $f_n(z^*)$ tends respectively to two values differing by $c \neq 0$. Let $\phi_r(z)$ be the difference $f_{n'_r}(z) - f_{n''_r}(z)$; as $r \to \infty$ $\phi_r(z)$ tends to zero at $z = z_m$ and to c at $z = z^*$. By Theorem 5, Cor. we can find a subsequence of (r) through which $\phi_r(z)$ tends to a limit function ϕ in D, and *uniformly* in any D'_-. By

a theorem of Weierstrass $\phi(z)$ is regular (in any D'_- and so) in D. Since $\phi(z) = 0$ at an infinity of points z_m in the neighbourhood of $z = z_0$ it must, by a classical theorem on the identity of two analytic functions, be identically zero, and this contradicts $\phi(z^*) = c$.

Thus $f_n \to f$ in D. Also f is continuous, as the limit of a uniformly continuous f_n ["$|\Delta f| = \lim |\Delta f_n| \leqslant \epsilon$ $(|\Delta z| < \delta)$"]. If now the convergence is not uniform in every D'_- there must exist a $k > 0$, an infinity of values of n, and corresponding points ζ_n with a limit point ζ in D, for which $d_n(\zeta_n) > k$, where $d_n(z) = |f_n(z) - f(z)|$. But since f is continuous and f_n uniformly continuous at ζ, $d_n(\zeta)$ and $d_n(\zeta_n)$ differ by arbitrarily little whenever ζ_n is within a distance δ (independent of n) of ζ. Therefore $d_n(\zeta) > \frac{1}{2}k$ for large n of the sequence; and this contradicts $f_n(\zeta) \to f(\zeta)$. Thus $f_n \to f$ uniformly in any D'_-. Hence finally, by Weierstrass's theorem, f is regular (in any D'_- and so) in D. Our results are therefore proved so far as they concern f_n. The results for the p-th derivative $f_n^{(p)}$ may be proved in the same way [the argument at (1) being available, with obvious modifications]. They may also be deduced by Weierstrass's theorem (since every D'_- is strictly interior to some other one) from those for f_n.

It would be convenient to have some short symbolism for "f_n converges to a regular f in D, and uniformly in any D'_-". "$f_n \to f$ uniformly in D" is open to the objection that it has already a meaning other than the one we intend (and incidentally a false one). [The objection is not absolutely fatal, since we practically never need to assert uniform convergence (in the ordinary sense) in an open set, and might alter its meaning without much fear of confusion.] A purely symbolical assertion like "$f_n \to f$ (V) in D" becomes uncomfortable when the fact has become very familiar that "$f_n \to f$ (V) in D" is a necessary consequence (by the theorem) of "$f_n \to f$ in D". We prefer to use merely "$f_n \to f$ in D", and, having once strongly directed the reader's attention to the point, expect him hereafter to read into the assertion all the consequences of it implied by the theorem.

10.61. *An alternative proof of a special case.*

Suppose (i) f_n *regular and uniformly bounded in* D, *a circle.* (ii) $f_n \to f$ *uniformly in* d', *a concentric smaller circle. Then* f *is regular in* D, *and* $f_n \to f$ *in* D, *uniformly in any* D'_-.

By hypothesis we have for $n, m > \nu(\epsilon)$,

$$|\phi| = |f_n - f_m| < \epsilon = M_1 \text{ (in } d').$$

Also
$$|\phi| < 2K = M_3 \quad (\text{in } D).$$

We may take D'_- to be $|z| \leqslant r_2 < r_3$, d' to be $|z| \leqslant r_1$, D to be $|z| < r_3$. Theorem 108 gives

$$M_2 \leqslant \epsilon^{\vartheta}(2K)^{1-\vartheta},$$

ϑ depending only on r_1, r_2, r_3. Since this tends to 0 with ϵ it follows that

$$|f_n - f_m| < \epsilon \text{ in } D'_- \text{ for } n, m > \nu'(\epsilon).$$

Hence there exists an f such that $f_n \to f$ uniformly in D'_-, and then f is regular in D'_-, by Weierstrass's theorem.

The general theorem also can be proved without the selection principle, but (if we are to take the simplest proof) not on these lines.

10.7. THEOREM 114 (*Montel*).—*Given a sequence* $\{f_n(z)\}$ *of functions regular in D and uniformly bounded in any* D'_-, *there exists a subsequence* $f_{n_\nu}(z)$ *and an* $f(z)$, *regular in D, such that* $f_{n_\nu} \to f$ *in D as* $\nu \to \infty$.

Take any z_0 in D, and a sequence $z_1, z_2, \ldots$ tending to z_0. The double sequence $f_n(z_m)$ is bounded; hence, by Theorem 5, we can select a subsequence f_{n_ν} such that, for each z_m, $f_{n_\nu}(z_m)$ converges to a limit as $\nu \to \infty$. The desired result follows by Theorem 113.

11. *Preliminary results on conformal representation.*

11.1. THEOREM 115.—*Suppose that* $f(z)$, *not a constant, is regular at* $z = a$, *and that* $f(a) = b$. *Then* f *takes, near* a, *any value near enough to* b. *More precisely, if* $z = a$ *is a zero of* $f(z) - b$ *of order* n *(exactly), then for every sufficiently small* s *there exists an* r, *tending to* 0 *with* s, *and such that for every* c *satisfying* $|c-b| \leqslant s$ *there are exactly* n *solutions of* $f(z) = c$ *in* $|z-a| < r$.

Suppose that $a = 0$, and that for small z

$$f - b = a_n z^n + a_{n+1} z^{n+1} + \ldots, \quad n > 0, \quad a_n \neq 0.$$

For all small r

$$|a_{n+1} z^{n+1} + \ldots| < \tfrac{1}{4}|a_n| r^n \quad (|z| = r).$$

If now $s = \frac{1}{4}|a_n| r^n$ and $|c-b| < s$, then $f - c = F + \phi$, where

$$F = a_n z^n, \quad \phi = (b-c) + (a_{n+1} z^{n+1} + \ldots),$$

and, on $|z| = r$, $\quad |\phi| < \frac{1}{2}|a_n| r^n, \quad |\phi/F| < \frac{1}{2} < 1.$

Round $|z|=r$ we have

$$\Delta \arg(f-c) = \Delta \arg F + \Delta \arg(1+\phi/F) = 2n\pi + 0.$$

Hence $f-c$ has n zeros in $|z|<r$.

Definition.—A function is called "schlicht" in D if $f'(z) \neq 0$ in D, and $f(z_1) \neq f(z_2)$ for distinct points z_1, z_2 of D. Or: if $f(z)-a=0$ has never more than one solution (counting multiplicities) for z of D.

THEOREM 115, COR.—*If f is regular at z_0, and $f'(z_0) \neq 0$, then there exists a neighbourhood of z_0 in which f is "schlicht".*

Here $n=1$. If z_1 is near z_0, the value $f(z_1)$ is near $f(z_0)$, and cannot be taken twice near z_0.

11.2. THEOREM 116.—*Suppose that f is regular and "schlicht" in a domain D. Then the values $w=f(z)$ "fill" a domain Δ. Also there is a function $\phi(w)$, the inverse of f, regular and "schlicht" in Δ, whose values z fill D. If D is simply connected, so is Δ. (Thus there is complete reciprocity.)*

If E is the aggregate of values w, and w_0 a point of E, it follows from Theorem 115 that all w near enough to w_0 belong to E. Hence E is an open set. E is connected, since if $w_0 = f(z_0)$, $w_1 = f(z_1)$ belong to E, so do the $w = f(z)$ corresponding to z of a polygon in D from z_0 to z_1, which w lie on a curve. Thus E is a domain Δ, possibly multiply connected.

If now w is a given point of Δ, there is a unique solution z in D of $w = f(z)$. We define

$$z = \phi(w) \quad (w \text{ in } \Delta),$$

and have to prove (since ϕ is certainly one-valued) that ϕ is differentiable in Δ; then evidently ϕ is regular and "schlicht", and its values fill D. Now Theorem 115 shows that ϕ is continuous at a w of Δ (for a value near w is *taken* by f near z, and can only be taken once at all). Hence if $z+\delta z$, $w+\delta w$ correspond by f, δz *tends to zero with* δw. Then

$$\frac{\delta z}{\delta w} = 1 \Big/ \frac{\delta w}{\delta z} \to \frac{1}{f'(z)} \quad \text{as} \quad \delta w \to 0$$

Thus ϕ is differentiable in Δ.

That Δ is simply connected follows from the one-one continuous correspondence with D.

When f, ϕ; D, Δ are (reciprocally) related as in Theorem 116, we say that $w = f(z)$ gives the "conformal representation of D on Δ", and $z = \phi(w)$ that of Δ on D.

COR.—*f need not be "schlicht". In this case, however, Δ must be taken on the appropriate Riemann surface; ϕ is multiform on the simple w-plane. Reciprocally f need not be uniform, if D is on a Riemann surface, and then ϕ is not "schlicht".*

[Omit the word "schlicht" in Theorem 116, and interpret "regular" in the usual conventions for Riemann surfaces. The details involve the usual treatment of branch points. Developments of this kind, however, we systematically omit.]

11.3. THEOREM 117.—*Suppose that C is a closed contour, D its interior; and that $f(z)$ is regular in D and continuous in D'. Suppose that, as z describes C in the positive direction, $w=f(z)$ describes a closed contour Γ once. Then* (1) *Γ is described positively, and* (2) *$w=f(z)$ gives a conformal representation of D on Δ, the interior of Γ.*

D and Δ are simply connected. After Theorem 116 it is enough to prove (1), together with

(3) Given z_0 in D, $w_0=f(z_0)$ lies in Δ;

(4) Given w_0 in Δ, there exists a z_0 in D, and *only* one, for which $w_0=f(z_0)$.

[For after (3) and (4) f is "schlicht", and its values fill Δ.]

Suppose z_0 is a point of D, and let $u=f(z_0)$. We have

$$\Delta_C \arg\{f(z)-f(z_0)\} = \Delta_\Gamma \arg\{w-f(z_0)\}. \tag{5}$$

The left is 2π times the number of roots of $f=f(z_0)$ in D, or at least 2π. Hence u cannot be an exterior point of Δ, or the right-hand side would be zero. If u were a frontier point of Δ, *i.e.* a point of Γ, $f(z)$ would take near z_0 all values near u, and so would take values that are exterior points of Δ, which we have seen to be impossible. Hence u is an interior point of Δ, (5) holds, and the right side is $\pm 2\pi$ according as Γ is described positively or negatively, while the left side is not less than $+2\pi$. Hence Γ is described positively. Thus (1) and (3) are proved.

Finally, if w_0 belongs to Δ (and so $f\neq w_0$ for z of C),

$$\Delta_C \arg\{f(z)-w_0\} = \Delta_\Gamma \arg(w-w_0) = +2\pi, \tag{6}$$

so that there exists one and only one z in D giving $f(z)=w_0$. This proves (4).

COR.—*In Theorem* 116, *to a closed contour C_1 lying in D, its interior, and its exterior* (*in D*), *correspond by the transformation respectively a closed contour Γ_1 in Δ, its interior, and its exterior; and C_1, Γ_1 are described in the same sense.*

The transform of C_1, having no double point, is a closed contour.

11 . 4. LEMMA 1.—*Let $f(z)$ be regular and "schlicht" in D. To a sequence (z_n) tending to z in D corresponds a sequence (w_n) tending to w in Δ. To a sequence (z_n) in D with one or more points of the frontier as limit points corresponds a sequence (w_n) in Δ with one or more points of the frontier as limit points (but the frontier points need not correspond point by point). To a closed D'_- corresponds a Δ'_-, frontiers corresponding point by point. Finally the distance $d\{\Delta'_-, F(\Delta)\}$ tends to zero with $d\{D'_-, F(D)\}$. All these results hold also reciprocally.*

The first part is obvious since $f(z_n) \to f(z)$, a point of Δ. The reciprocal of this, and the reciprocals of all other proved results, are, of course, immediate.

If the second part is false, then every subsequence of the w_n tends to a w interior to Δ, and by the reciprocal of the first part the z-subsequence corresponding (therefore an *arbitrary* one) tends to a z interior to D, which is false.

In the third part, to (the interior of) D_- corresponds a domain Δ_1, and to the frontier of D_- must correspond, point by point, that of Δ_1; since, if z_0 belongs to $F(D_-)$, $f(z)$ takes, near z_0, values belonging to Δ_1, and others not belonging to Δ_1; and reciprocally. Finally, every point of $F(\Delta_1)$ being interior to Δ, $\Delta'_1 = \Delta_1 + F(D_1)$ is interior to Δ.

If the fourth part is false, there exists a sequence D'_n with $d_n = d\{D'_n, F(D)\}$ tending to zero, and a corresponding sequence Δ_n such that each $F(\Delta_n)$ contains a w_n distant more than $d > 0$ from $F(\Delta)$. The w_n have some limit-point w, necessarily *interior* to Δ. But then, by the reciprocal of the first part, the corresponding z_n have a limit-point z of D, and this contradicts $d_n \to 0$.

NOTE.—It is not proved, nor is it true, that if z tends to a z on $F(D)$ then w tends to a (unique) w on $F(\Delta)$.

11 . 51. Theorem 117 is important in applications in which we are given D and Δ and have to find an f; it is enough to find a regular f that behaves correctly on the boundary C.

The domains we have been considering are bounded; we often require, however, the function representing a given D on a half-plane. Following the usual rules of thumb we might expect the following modification of Theorem 117 to hold (and it is often naively appealed to by mathematical physicists):

(A) *Suppose that C is a closed contour starting from and ending at $z = a$, and that $f(z)$ is regular inside C, and continuous on C except at $z = a$, while $f \to \infty$ (uniformly) as $z \to a$ in D. Suppose further that*

as z describes C, $w = f$ describes the real axis from $-\infty$ to $+\infty$. Then $w = f$ conformally represents D on Π, the upper half-plane of w.

This proposition, however, is simply false. Let $f(z) = i(1+z)/(1-z)$; $w = f(z)$ represents $|z| < 1$ on Π, and as z describes $|z| = 1$, starting with $z = 1$, f increases steadily from $-\infty$ to ∞. Consider now $\phi = f^3$. ϕ increases steadily from $-\infty$ to ∞ as z describes the circle, ϕ is continuous on the circle except at $z = 1$, and $\phi \to \infty$ as $z \to 1$. But if $\mathfrak{I} w_0 > 0$ two of the cube roots of w_0 lie in Π and are values of f, so that the value w_0 is taken twice by ϕ (in γ); and if $\mathfrak{I} w_0 < 0$, *one* cube root lies in Π and ϕ takes the value w_0 once.

It is desirable, of course, to have a true form of (A) with the minimum of extra hypotheses. We give three such forms.

11 . 52. *Sufficient extra conditions under which* (A) *is true.*

(1) *There exists a w_0, not real, such that $f \neq w_0$ in D.* (We shall see that $\mathfrak{I} w_0$ is necessarily negative.) It is easily verified that (if w_0 is not real) $\zeta = 1/(w - w_0)$ describes a certain circle Γ (interior Δ) as w goes from $-\infty$ to ∞, that the transformation represents Δ on that half-w-plane in which w_0 does not lie, and that Γ is described positively or negatively according as w_0 does not or does lie in Π. Consider now

$$\zeta = \phi(z) = 1/\{f(z) - w_0\}.$$

It is regular in D, and continuous in D' (including $z = a$). As z describes C, f describes $-\infty$ to ∞, and ζ describes Γ, once. It follows by the main theorem that $\zeta = \phi(z)$ represents D on Δ. Also that Γ is described positively; so *w_0 is not in Π*. Then, combining the transformations, we see that $w = f(z)$ represents D on Π.

In the remaining cases we suppose, but purely for simplicity, that in the neighbourhood of $z = a$, C consists of two analytic curves C', C'' meeting at a.

Suppose pq is an arc of a small circle with a as centre, joining C' and C'', and lying, except for p, q, in D. Then (A) is true if :

(2) *As z describes a pq near a, f describes a curve lying in* Π, *or more generally, describes a curve not cutting a fixed curve Λ, where Λ "extends to ∞", and never cuts the real axis of w.* [Actually Λ must lie in Π, though we need not assume this explicitly.]

Denote by C_1 the contour C modified by the cutting out of a by pq, and let ξ be a point of Λ. As z describes C_1, $f(z)$ describes a closed (but not necessarily *simple*) curve Γ_1, *not cutting* Λ. Now $\Delta_{C_1} \arg (f - \xi) = \Delta_{\Gamma_1} \arg (w - \xi) = 0$ for sufficiently distant ξ. Also the

left side is a continuous function of ξ except for ξ of Γ_1, and is also of the form $2n\pi$ except for ξ of Γ_1. It is therefore zero for all ξ, since ξ never lies on Γ_1. Hence, for any fixed ξ, $f \neq \xi$ in D_1 Since D_1 differs arbitrarily little from D, $f \neq \xi$ in D. We may therefore apply case (1).

[NOTE.—We cannot, in the special case, argue directly that $w = f$ represents "D_1" on a "Π_1" and take the limit, since Γ_1 may not be simple.]

Finally: (A) is true if:

(3) *f is defined in a complete neighbourhood of a and satisfies*

$$f \sim c(z-a)^{-\lambda} \;\; (\tfrac{1}{2} < \lambda < \tfrac{3}{2}) \quad as \quad z \to a;$$

in particular it is true if f has a simple pole at $z = a$.

Suppose $a = 0$, and let $\arg(z-a) = \arg z = \theta$. Then

$$\arg\{(z-a)^\lambda f\} = \lambda \arg z + \arg f$$

is, near a, $\lambda\theta + (2m+1)\pi$ on C_1 and $\lambda\theta + 2n\pi$ on C_2. Since these must tend to limits $\arg c + 2p\pi$ we see, (1) that C_1 and C_2 must have tangents at a (which, of course, we are already assuming), and (2) the angle α formed at a has a magnitude $(2k+1)\pi/\lambda$. Since $\alpha \leqslant 2\pi$ and $\lambda < \frac{3}{2}$ we must have $k = 0$, $\alpha = \pi/\lambda$. Since further $\lambda > \frac{1}{2}$, α is positive and *less* than 2π. We draw a piece of a straight line L bisecting the complementary angle at a. Now for any x of L and any z of D

$$\underline{\lim}\,|\arg f(z) - \arg f(x)| \geqslant \lambda(\pi - \tfrac{1}{2}\alpha) = (\lambda - \tfrac{1}{2})\pi > 0$$

as $x, z \to a$. Hence $f(x) \neq f(z)$ if x and z are confined to a circle of some radius r round $z = a$, to which we suppose L confined. Then, in the notation of case 2, the transform of pq does not meet the transform Λ of L. Also for distant w of Λ we have $\arg w = \pm\frac{1}{2}\pi$ nearly; hence Λ, beyond some point, does not meet the real axis of w. We can now apply case 2.

11.53. Another problem presents itself: suppose $z = a$ is replaced by $z = \infty$. *This*, however, is trivial. We have

THEOREM 117, COR. 2.—*Suppose C is a curve, extending to ∞ at both ends and simple, and let D be one of the domains into which C divides the plane. Suppose f is regular in D and continuous in D' (except for $z = \infty$) and $f \to l$ as $z \to \infty$ in D; further that, as z describes C, $w = f(z)$ describes a (bounded) closed contour Γ once. Then $w = f(z)$ represents D on Δ, the interior of Γ.*

There exist points $z = \alpha$ exterior to D. Then $z_1 = 1/(z-\alpha)$ trans-

forms C (plus the point $z = \infty$) and D into a bounded closed contour C_1 and its *interior*† D_1 and $f(z)$ into $f_1(z_1)$ say. By the main theorem $w = f_1(z_1)$ represents D_1 on Δ.

12. *The theory of the linear function*

$$= L(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0.$$

12.1. This theory is important in the sequel, and we break off to give a systematic account of it. ζ is regular except at $z = -d/c$ (and not excepting $z = \infty$). Also ζ is "schlicht" in any domain. The inverse function

$$z = \frac{d\zeta-b}{-c\zeta+a}$$

is regular except at $\zeta = a/c$. If we make $z = \infty$ and $\zeta = a/c$; $z = -d/c$ and $\zeta = \infty$ correspond we have a one-one correspondence between the z- and ζ-planes. (If $c = 0$ the points ∞ correspond.)

Let z', ζ'; z'', ζ'' correspond. The equation is then

$$\text{(A)} \qquad \frac{z-z'}{z-z''} = m\frac{\zeta-\zeta'}{\zeta-\zeta''}, \quad m \text{ a constant.}$$

In what follows we shorten the discussion by appeal to geometry (see Fig. 6). Let z_1, z_2, z_3, z_4 be four distinct points of a circle. Then

$$\frac{z_3-z_1}{z_3-z_2} = \frac{r_{31}}{r_{32}}e^{i\alpha}, \quad \frac{z_4-z_1}{z_4-z_2} = \frac{r_{41}}{r_{42}}e^{i\beta}.$$

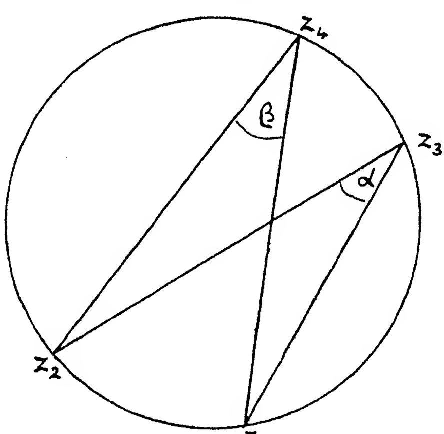

Fig. 6.

† Interior *or* exterior, and the latter is impossible since ∞ corresponds to $z = z_0$.

Since $\alpha=\beta$ or $\pi+\beta$

$$\frac{z_3-z_1}{z_3-z_2}\Big/\frac{z_4-z_1}{z_4-z_2}=\lambda$$

is real, and $\lambda \neq 1$ if z_2, z_4 are distinct.

Conversely, if the cross-ratio is real the four points are concyclic. Now the cross-ratio is the same as that of the corresponding ζ. Hence z-circles correspond to ζ-circles, and conversely (straight lines are regarded as circles).

Let us now represent ζ-points on the z-plane. (We then speak of "invariant" points, curves, families of curves.)

12.2. *The fixed points of the substitution.* These are given by $\zeta=z$ or

$$cz^2+(d-a)z-b=0.$$

with roots z_1, z_2 say. If $a=d$, $b=c=0$, we have the identical substitution; *every* point is fixed. Rejecting this there are two roots or a double root, reckoning $z=\infty$ as a solution if $c=0$.

Suppose first z_1, z_2 finite and distinct. We denote by K a circle through z_1, z_2; by K' a circle with z_1, z_2 as inverse points. The K's go into circles through z_1, z_2. The sheaf of K's therefore transforms into itself. By the angle-properties of a conformal transformation the K''s, being orthogonal to all K's, transform into circles orthogonal to all K's, *i.e.* into circles K'. The sheaf of K''s is invariant.

We have now to distinguish three cases.

1. *Every K transforms into itself* (not, of course, point by point). The intersection of a K' and a K transforms into the intersection of the new K' with the *same* K. We may think of the transformation taking place by each point moving along the K through it into its new position. The K's are the "tracks" of the transformation†. This type of transformation is called hyperbolic.

2. *Elliptic.* *Every K' goes into itself.* The K''s are the tracks.

3. *Loxodromic.* The general case: neither (1) nor (2).

Normal forms of the transformation. Let z and z_3 be two points on a K, $z_3 \neq z_1, z_2$. Then

$$\text{(1)}\qquad \frac{z-z_1}{z-z_2}\Big/\frac{z_3-z_1}{z_3-z_2}=\frac{\zeta-\zeta_1}{\zeta-\zeta_2}\Big/\frac{\zeta_3-\zeta_1}{\zeta_3-\zeta_2}=\frac{\zeta-z_1}{\zeta-z_2}\Big/\frac{\zeta_3-z_1}{\zeta_3-z_2}.$$

† The transformation depends on a continuous parameter a (besides z_1, z_2). As this is *varied* the tracks come into being.

In case 1, z_1, z_2, z_3, ζ_3 lie on a K. Hence

$$\frac{z_3-z_1}{z_3-z_2}=a\frac{\zeta_3-z_1}{\zeta_3-z_2} \quad (a \text{ real}, \neq 1).$$

Hence the normal form is

$$\frac{z-z_1}{z-z_2}=a\frac{\zeta-z_1}{\zeta-z_2} \quad (a \text{ real}, \neq 1).$$

Conversely if a is real and $\neq 1$, this equation makes z, ζ, z_1, z_2 lie on a K.

Case 2.—Here z and ζ lie on a K'. Therefore

$$\left|\frac{z-z_1}{z-z_2}\right|=\left|\frac{\zeta-z_1}{\zeta-z_2}\right|,$$

$$\frac{z-z_1}{z-z_2}=e^{i\phi}\frac{\zeta-z_1}{\zeta-z_2} \quad (\phi \text{ real}).$$

The converse also holds.

Case 3.—
$$\frac{z-z_1}{z-z_2}=\mu e^{i\phi}\frac{\zeta-z_1}{\zeta-z_2}$$

$$[\mu>0; \quad \phi \not\equiv 0 \pmod{2\pi} \text{ and } \mu\neq 1].$$

Suppose next $z_2=\infty$, z_1 *finite.* The forms become

(1) $z-z_1=a(\zeta-z_1)$ (a real, $\neq 1$). ("Expansion" about z_1.)

(2) $z-z_1=e^{i\phi}(\zeta-z_1)$ (ϕ real). (Rotation about z_1.)

(3) $z-z_1=\mu e^{i\phi}(\zeta-z_1)$ ($\phi\not\equiv 0$ and $\mu\neq 1$).

In (3) the invariant curves are logarithmic spirals (loxodromes). For suppose $z_1=0$, $\mu e^{i\phi}=e^{p+iq}$. The curves $z=e^{a+(p+iq)t}$ $(-\infty<t<+\infty)$ are invariant (if a is a real parameter). For if $\zeta=e^{a+(p+iq)T}$ the transformation becomes $t=T+1$.

Finally suppose $z_1=z_2$. This substitution is called *parabolic.*

(a) z_1 finite. The K's become circles with a common tangent at z_1, and the K''s circles with a common tangent at z_1 normal to the first. The normal form is

$$\frac{1}{z-z_1}=\frac{1}{\zeta-z_1}+b \quad (b\neq 0).$$

(b) If $z_1=\infty$, the K and K' are orthogonal sets of parallel lines. The transformation is

$$z=\zeta+b' \quad (b'\neq 0),$$

a translation.

12.3. The general equation of linear transformation contains three arbitrary *complex* constants, and we can transform any three points (and their circle) into any three points (and their circle).

Inversion.—The points ζ and z are inverses in the unit circle if $\bar{\zeta} = 1/z$. The transformation $\bar{\zeta} = 1/z$ (from z to ζ) is a "conformal transformation with reversal of angle".

LEMMA 2.—*If z, z' are inverses with respect to K, then for a linear transformation ζ and ζ' are inverses with respect to K', the transform of K.*

For K and the system of orthogonal circles to it through z and z' transform into K' and the orthogonal circles to it through ζ and ζ'.

Inversion in a straight line is a reflection.

EXAMPLE.—*To find all linear transformations of $\mathfrak{I}z \geqslant 0$ into $|\zeta| \leqslant 1$.*

The inverse of z in the real axis is $\bar{z}$, that of ζ in the unit-circle is $1/\bar{\zeta}$. If

$$\zeta = \frac{az+b}{cz+d} = \frac{a}{c}\,\frac{z-\beta}{z-\gamma}$$

is the transformation, then $z = \beta$, γ correspond to $\zeta = 0$, ∞, inverses in the circle. Therefore β, γ are inverses in the line; $\gamma = \bar{\beta}$. Also $z = 0$ must go into a point of the unit ζ-circle. Hence $a/c = e^{i\tau}$ (τ real). Thus the transformation must be

$$\zeta = e^{i\tau}\frac{z-\beta}{z-\bar{\beta}}. \tag{1}$$

Finally $z = \beta$ goes into $\zeta = 0$, and β must belong to the *upper* half-z-plane.

Conversely, if $\mathfrak{I}\beta > 0$ (1) does what is required. For $|\zeta| = 1$ for z of the real axis, and $\arg\zeta$ increases by $+2\pi$ as z goes from $-\infty$ to ∞: the result follows by Theorem 116, Cor.

There are three *real* constants implied in τ, β, which we can use to make three points of the line and three of the circle correspond. If the triplets have the same sense $\arg\zeta$ must increase by 2π as z describes the real axis positively, we must have $\mathfrak{I}\beta > 0$, and $\mathfrak{I}z > 0$ corresponds to $|\zeta| < 1$. If they have opposite senses then $\mathfrak{I}\beta < 0$, and the lower half z-plane corresponds to the ζ-circle. We can also make $\zeta = 0$ correspond to a given β ($\mathfrak{I}\beta > 0$) and a direction OT at the centre of the circle to a direction at β.

To find all linear transformations of a circle (and the interior) into itself.

Let the circle be $|z| \leqslant 1$. Write the general transformation

$$\zeta = \gamma \frac{z-a}{\beta z - 1}.$$

a must be in the circle. Since $\zeta = 0$, $\zeta = \infty$ must have an inverse pair as correspondents, we have $\beta = \bar{a}$. For $z = 1$ we must have $|\zeta| = 1$, hence

$$1 = |\gamma| \left| \frac{1-a}{\bar{a}-1} \right| = |\gamma|.$$

Hence the most general transformation is

$$(2) \qquad \zeta = e^{i\tau} \frac{z-a}{\bar{a}z - 1} \quad (\tau \text{ real}, \ |a| < 1).$$

12.4. LEMMA 3.—*The transformation* (2) *transforms the interior of the unit-circle into itself, and the exterior of the unit-circle into itself.*

We use inversion, and the last result.

It follows without difficulty that we can, in a linear transformation of the unit circle (and interior) into itself, make three points of the boundary correspond to three others, provided the triplets have the same sense; or we can make one internal point and a direction at it correspond arbitrarily, and either condition determines the transformation uniquely.

12.5. We end this section by proving

THEOREM 118. (Generalization of the symmetry principle).—*Suppose two arcs of circles AB, $a\beta$ are free portions of the boundary of domains D, Δ. Let $f(z)$ be regular in D and continuous (in D') on AB, and let $\zeta = f(z)$ represent D on Δ, AB, $a\beta$ being corresponding arcs. Let D^*, z^* be the inverses of D, z in (the circle of) AB, and Δ^*, ζ^* those of Δ, ζ in $a\beta$. If now $\zeta = f(z)$ (for z of D) gives $\zeta^* = \phi(z^*)$, say, then $\phi(z^*)$ is a regular function of z^* for z^* in D^*, and is the continuation of f across (AB); f is further regular at points of (AB).*

If $v = f(z)$ represents a domain Z on V, and $w = \phi(v)$ represents V on W, then $w = \phi\{f(z)\}$ represents Z on W.

By linear transformations we can turn AB into the real axis of z_1, and $a\beta$ into the real axis of ζ_1. Then z^* (*qua* z-point) becomes $\bar{z}_1$, ζ^* becomes $\bar{\zeta}_1$, and $\zeta = f(z)$, $\zeta^* = \phi(z^*)$ become $\zeta_1 = f_1(z_1)$, $\bar{\zeta}_1 = \phi_1(\bar{z}_1)$. The functions f_1, ϕ_1 are related as in the original Theorem 105, and we obtain the general form by transforming back.

It is the "symmetrical" *value* of the function when continued that gives this theorem its great importance. The rather different question of the conditions under which *some* continuation is possible is, however,

also important, and can be answered completely. We therefore discuss it here (though it is alien to our present ideas).

COROLLARY. *Suppose that a free arc AB of the boundary of D is an analytic arc without singular point, that $f(z)$ is regular in D, continuous (in D') at points of AB, and that $w = f(z)$ lies upon an analytic arc without singular point in the w-plane as z describes AB. Then $f(z)$ can be continued across AB.*

An analytic arc AB without singular point is a curve for which, in some parametric representation $z = z(t)$ (t real, $t_1 \leqslant t \leqslant t_2$), $z(t)$ is a regular function of t, and $z'(t) \neq 0$, at all points of the interval (t_1, t_2). Consider any point z_0 of AB; we may suppose that this corresponds to the value $t = 0$. By Theorem 115, Cor., the transformation $z = z(t)$ transforms the neighbourhood of z_0 into a neighbourhood of $t = 0$ in the t-plane, and must therefore transform the piece $A'B'$ of AB included in the neighbourhood into an interval of the real axis of t. Similarly the transformation $w = w(\tau)$ (associated with the parametric representation of the w-arc $\alpha\beta$) transforms the corresponding piece of $\alpha\beta$ into an interval of the real axis of τ. Further [since $z'(t) \neq 0$, and $w'(\tau) \neq 0$ near z_0, w_0], both transformations have (regular) inverse transformations, $t = t(z)$, $\tau = \tau(w)$ respectively in the neighbourhoods of z_0, w_0. To a neighbourhood of z_0 bounded (partly) by $A'B'$ corresponds a neighbourhood of $t = 0$ bounded by a piece of the real axis, and similarly for $\alpha'\beta'$ and the τ real axis. The equation $\tau = \tau[f\{z(t)\}] = \phi(t)$ makes $\phi(t)$ real for small real t; hence, by the original symmetry principle (Theorem 105), $\phi(t)$ can be continued across the real axis near $t = 0$. It follows that

$$f(z) = w[\phi\{t(z)\}]$$

can be continued across AB near $z = z_0$.

13.1. We resume now, after the interruption of section 12, our discussion of conformal representation in general.

THEOREM 119.—*If the conformal representation of a simply-connected D on a circle d is possible at all, it can be effected with a given point and direction in D corresponding to a given point and direction in d; or, if the representation is one of D' on d', the boundaries, supposed to be closed contours, corresponding point by point, it can be effected with three points of the boundaries (the same way round) corresponding arbitrarily. Further, either of these conditions uniquely determines the representation.*

COROLLARY.—*A conformal transformation of a circle into itself is necessarily linear.*

The first part is now obvious: the new representation can be made *via* a linear transformation of the circle into itself.

If two distinct transformations exist with the same set of conditions, we obtain (by combining one with the inverse of the other) a non-identical transformation of the circle into itself with an invariant centre and direction there, or three invariant boundary points. This is impossible. Suppose first that $\zeta = f(z)$ transforms the unit circle into itself, with $f(0) = 0$, $f'(0) > 0$. Since $|f(z)| < 1$ for $|z| < 1$, and $f(0) = 0$, we have

$$|\zeta/z| = |f/z| \leqslant 1 \quad \text{(Theorem 107).}$$

In particular $(z = 0)$ $|f'(0)| \leqslant 1$. But $z = g(\zeta)$, the inverse transformation, is of the same type; hence $|1/f'(0)| = |g'(0)| \leqslant 1$. Hence $|f'(0)| = 1$, $f'(0) = 1$. But now $|f(z)/z|$ takes at $z = 0$ its upper bound in $|z| < 1$. Hence, by Theorem 101,

$$f(z)/z = \text{constant} = a_1 = 1.$$

If, on the other hand, a non-identical transformation T of the unit-circle into itself leaves three points of the circumference invariant it must, by the above, change the position of $z = 0$ or the direction of a line through it. The linear transformation L that restores them changes the boundary triplet (having only two fixed points), so TL is not identity. But TL is of the type just considered, and we have a contradiction.

To deduce the corollary we need only observe that there exists one transformation, and a linear one, that transforms the centre and a direction there in the same way as the given one.

THEOREM 120.—*The representation of a bounded ("schlicht"†) domain D (not necessarily simply connected) on a domain Δ is, if possible at all, uniquely determined by the correspondence of a point and direction in D with a point and direction in Δ.*

The proof of the special case does not extend, and we must start afresh. It is enough to prove:

If $w = f(z)$ represents a bounded "schlicht" D, containing the origin, on itself, with $f(0) = 0$, $f'(0) = a > 0$, then $f(z) \equiv z$.

Let C be a circle, radius r and centre $z = 0$, containing only points of D. Write f_{-1} for the inverse of f, $f_n = f(f_{n-1})$, $f_{-n} = f_{-1}(f_{-(n-1)})$, $f_0 = z$: all these have the same domain D of existence, and

$$|f_n| \leqslant M = \operatorname*{Max}_{(D)} |z|.$$

† A "schlicht" domain is one that does not overlap itself.

Hence, by Cauchy's theorem,

(1) $$|f_n^{(p)}(0)| \leqslant p!\ Mr^{-p}.$$

Now it is easily seen that $f_n'(0) = a^n$ (for n of either sign): (1) (with $p=1$) therefore requires $a=1$. But then, unless $f \equiv z$, we have in C

$$f = z + a_p z^p + \ldots \quad (a_p \neq 0),$$

and so $$f_n = z + n a_p z^p + \ldots,$$

which again is incompatible with (1).

COR. 1.—*If $w = f(z)$ represents a bounded "schlicht" domain D, containing the origin, on a domain Δ, and if $f(0) = 0$, $f'(0) = 1$, then either f is z and $\Delta = D$, or else D and Δ overlap in the strict sense, i.e. they have common points, and each contains points not belonging to the other.*

Suppose that $\Delta \subset D$. Then, on the one hand

$$f_1(z) = f, \quad f_2(z) = f\{f(z)\}, \quad \ldots$$

evidently exist in D and are bounded by M, as in the main theorem; and on the other $f_n = z + n a_p z^p + \ldots$ for positive n and small z (also as before), if $f \not\equiv z$. Hence $\Delta \subset D$ implies $f \equiv z$, and similarly $D \subset \Delta$ implies $f_{-1} \equiv z$, or $f \equiv z$.

COR. 2.—*If $w = f(z)$ represents D, containing $z = 0$, on a Δ contained in D, and if $f(0) = 0$, then $|f'(0)| \leqslant 1$.*

For $$|f'(0)|^n = |f_n'(0)| \leqslant Mr^{-1}.$$

13.2. We recall that any one-one *continuous* correspondence between two domains preserves the connectivity. A closed contour and its interior become a closed contour and its interior (or, with some conventions about infinity, possibly the exterior).

LEMMA 4.—*A simply-connected D (in a single-sheeted surface), with more than one point in its frontier, can be conformally represented on some bounded domain.*

If z_1 and z_2 belong to $F(D)$ we may suppose $z_2 = \infty$: otherwise take $Z = 1/(z-z_2)$ (a "*schlicht*" transformation). Then infinity is not interior to D. Let $\zeta = \sqrt{(z-z_1)}$. A closed contour in D cannot surround z_1, hence ζ is regular in D. $\zeta(z)$ is also "schlicht" (since $z = z_1 + \zeta^2$). Suppose then that $\zeta = \zeta(z)$ transforms D to Δ. If ζ_0 (necessarily $\neq 0$) is an interior point of Δ, $-\zeta_0$ is *not* one, otherwise z_0 in D has two correspondents $\pm\zeta_0$ in Δ. Since ζ *is* interior, and $-\zeta$ is *not*, also when ζ is *near* ζ_0, it follows further that $-\zeta_0$ is not a point of

$F(\Delta)$ (if it were it would have neighbouring $-\zeta$ that were interior). Hence $-\zeta_0$ is an exterior point of Δ. If now $Z = 1/(\zeta+\zeta_0)$ we obtain a Z-domain that is bounded.

This result can be pushed further. (We do not actually use the extension *for conformal representation theory.*)

LEMMA 5.—*Suppose that D is a bounded simply connected domain, and that P is one of its frontier points. Then D can be conformally represented on a Δ interior to a finite circle K, and in such a manner that P corresponds to a point P_1 of the circumference of K, that is, all points of D near P become points of Δ near P_1. If D is bounded by a closed contour, so will Δ be.*

We may suppose DP to be $z=0$. Let $\underset{(D)}{\text{Max}}|z| = R$. Now let $\zeta = \log z$, a function regular in D. This transforms D into Δ, lying in the half-plane Π, or $\mathfrak{R}\zeta \leqslant \log R$. Also points near P in D go into points near ∞ in Δ. If now we transform Π into the interior of a ζ-circle K we arrive at our conclusion.

Note on the condition for D in Lemmas 4 and 5.—For a single (or finite) sheeted surface an $F(D)$ that contains two points must contain at least a connected continuum extending to infinity if D is to be simply connected. *But this is not true if there may be an infinity of sheets.*

The excepted case.—Suppose D is bounded by one point. Then D cannot be conformally represented on a domain bounded by more than one.

For, by Lemma 4, if it were we could represent it on a bounded domain. Since we may also suppose the missing point to be $z=\infty$, we should have a representation $\zeta = \phi(z)$, with ϕ regular for all finite z, not constant, and bounded. This is impossible.

Finally we can show :

The only transformations that transform one D bounded by one point into another are linear (and then the missing points also correspond).

If not, we could, by combining with a suitable linear transformation, transform the finite plane, *non-linearly*, into itself. The transformation and its inverse are

$$\zeta = f(z), \quad z = g(\zeta),$$

where f and g are regular at all finite points, *and "schlicht"*. Hence f is not a *polynomial* of degree > 1, and so has an essential singularity at infinity. Then ζ takes arbitrarily small values for some large values of z, and this is inconsistent with $z = g(\zeta)$.

14. *Two lemmas about domains.*

14.1. In what follows we confine ourselves to simply-connected domains, on a single-sheeted plane, and not overlapping themselves ("schlicht"), for which $F(D)$ contains more than one point.

LEMMA 6.—*Suppose that we are given an expanding sequence of simply-connected domains D_n, bounded in their ensemble. Let $D = \Sigma D_n$. Then* (i) *D is a bounded simply-connected domain,* (ii) *$D_n \subset D$,* (iii) *$D'_- \subset D_n$ $(n > n_0)$.*

(ii) is trivial. (iii) practically *is* the generalized Borel theorem. (Every point of D is interior to some D_n, therefore a finite number of D_n cover D'_-, so therefore does the greatest of these.) It follows further that D is connected (if z_1, z_2 belong to D they belong to a D_n). Also D is open and bounded; hence it is a bounded domain. Finally if C is a closed contour of D, C, being a closed set of points, lies in a D_n, therefore *contains* only interior points of D_n, therefore only interior points of D. D is simply connected. This ends the proof.

[In the language of "limit-sets", $D_n \to D$.]

14.2. LEMMA 7.—*Given a simply-connected bounded D, there exists a strictly expanding sequence D_n of simply-connected domains bounded by polygons with sides parallel to the axes, and a sequence l_n, such that* (a) *D is the sum* (*or limit*) *of either the D_n or the D'_n, so that also $D'_- \subset D_n$ for large n;* (b) *any two points of D'_n can be joined, in D'_n, by a polygon of length $\leqslant l_n$.*

Cover the plane with a network of sides $2^{-1}, \ldots, 2^{-n}, \ldots$. The meshes of order n interior to D coalesce into distinct pieces, which are simply-connected (otherwise the outer boundary of the piece would surround frontier points of D). Call D'_n the piece containing a fixed internal point z_0 of D. In the first place any point z of D belongs to D_n for $n > n_0(z)$. For join z_0 to z in D, and let δ be the distance of the path from $F(D)$; it is easily seen that z belongs to D'_n if $2^{-n} < \frac{1}{4}\delta$. Hence $D = \Sigma D_n$ (since evidently $D_n \subset D$). Since $D'_n \subset D$, we have $D'_n \subset D_m$ for some m (Lemma 6), and so also $D = \Sigma D'_n$. For the same reason we can, by taking an appropriate subsequence, secure that $D'_{n_m} \subset D_{n_{m+1}}$ for all m, so that D_{n_m} is *strictly* expanding. Finally if M is the diameter of D, any two points of D'_{n_m} can be joined by a polygon of length $\leqslant 2M +$ perimeter of D_{n_m}.

15.1. LEMMA 8.—*Suppose f_n regular and $f_n \to f$ in D. Then $f_n - c$ and $f - c$ have the same number of zeros in D_- for $n > n_0$ (c, D_-)*

(counting multiplicity), provided no zero of $f-c$ lies in $F(D_-)$. (If a zero does so lie the result may be false). The provision requires in particular that $f-c$ must not be identically zero.

$f-c$ has a finite number of zeros in D'_-. Surround these by (small) circles lying in D_-. In the rest of D'_- (a closed set if the circles are taken open), $f-c$ is continuous and never zero, therefore bounded below in absolute value, by 2δ, say. By Theorem 113 we can choose n_0 so that

$$|f_n-f| < \delta \quad (n \geqslant n_0, \ z \text{ in } D_-).$$

Then evidently $f_n - c \neq 0$ in D'_- outside the circles, and, since *on* a circle

$$\left|\frac{f_n-f}{f-c}\right| < \tfrac{1}{2},$$

f_n-c, or $(f-c)+(f_n-f)$, has as many roots inside as $f-c$. This proves the result.

COR.—*If each f_n is "schlicht" in its domain of existence, then either f is "schlicht" in D, or else f is a constant.*

If f is not constant and takes the value c more than once, there exists a D'_- in which it has the same property, while $f \neq c$ in $F(D'_-)$. By the lemma f_n-c has two zeros in D'_- for large n, and this is false.

16. *Riemann's existence theorem.*

16.1. We come now to the fundamental theorem on conformal representation (of a "schlicht" domain).

THEOREM 121.—*A simply-connected "schlicht" domain containing $z=0$, whose frontier contains more than one point, can be conformally represented on $|\zeta| < 1$ by $\zeta = f(z)$, with $f(0)=0$, $f'(0)$ real and positive.*

We may suppose the domain bounded, by Lemma 4; also translated and reduced in scale ($z' = az+b$). It is therefore enough to solve the problem of representing D on d with $f(0)=0$, $f'(0)>0$, where d is the unit ζ-circle, D contains $z=0$, and $D' \subset d$.

We represent z- and ζ-points in the same plane.

Our main idea is to seek a transformation $\zeta_1 = f_1(z)$, regular in D, which increases the distance of each point of D from 0 (makes $|\zeta_1| > |z|$), but leaves it in d. D then goes into a larger D_1. Similarly D_1 into D_2, and so on. We may hope to secure that $D_n \to d$, $f_n \to f$, and that f will do what we want.

Consider

$$(1) \qquad \frac{Z-\sqrt{r}}{\sqrt{r}Z-1} = \sqrt{\left(\frac{z-r}{rz-1}\right)} \quad (r<1, \ \sqrt{\ } = +r \text{ at } z=0).$$

Z is "*schlicht*", regular in any simply-connected domain Δ contained in d and containing $z=0$ but not (as an *interior* point) $z=r$, and continuous in Δ'.

As a function of Z z is regular in $|Z|<1$ and continuous in $|Z|\leqslant 1$; also $z=0$ corresponds to $Z=0$. Now if $|\zeta_0|\leqslant 1$, $|(\zeta-\zeta_0)/(\zeta_0\zeta-1)|\leqslant 1$ if and only if $|\zeta|\leqslant 1$, and the signs of equality correspond. It follows from (1) that $|z(Z)|\leqslant 1$ if $|Z|\leqslant 1$, and that $|Z(z)|\leqslant 1$ if $|z|\leqslant 1$ whichever sign is given to the square root; also that signs of equality correspond. By Theorems 107, 101, since z/Z is evidently not a constant,

$$\left|\frac{z}{Z}\right|<1 \quad (|Z|<1),$$

from which we have, respectively for $Z=0$ and $Z\neq 0$,

$$\left|\frac{dz}{dZ}\right|_{Z=0}<1, \tag{2}$$

$$|z|<|Z| \quad (0<|Z|<1). \tag{3}$$

The inverse function $Z(z)$ is regular in Δ and continuous in Δ', and is defined at $z=0$; and for z of Δ' we have [by (3) and (2)],

$$1>|Z|>|z| \quad (|z|<1), \tag{4}$$

$$|Z'(0)|>1. \tag{5}$$

16.2. We observe concerning the transformation (1):

1. When Z describes $|Z|=1$, z describes $|z|=1$ *twice*.

2. It is not possible for a function Z, *regular* (and not constant) in $|z|\leqslant 1$, to have $|Z|=1$ on $|z|=1$ and $|z|<|Z|$ for $0<|z|<1$, and we cannot avoid many-valued functions altogether. For such a Z must vanish *somewhere* (Theorem 101, Cor. 3), and this can happen only at $z=0$. Then we should have

$$|Z/z|\leqslant 1 \quad (|z|<1),$$

contrary to hypothesis.

3. We have so far given an account that does not mention Riemann surfaces. But the wider point of view is the right one, and suggests quite naturally the transformation (1), to whose origin there has so far

been no clue. (This is not the only occasion when Riemann surfaces suggest a proof of a theorem with which they have *prima facie* nothing to do. Compare § 25 . 31.)

Suppose that we represent a Riemann surface of two sheets, bounded in each sheet by $|z|=1$ and branched at $z=r$, on $|Z|\leqslant 1$ [so that $z(Z)$ is regular], making $z=0$, in *some* sheet, and $Z=0$ correspond. Then $z(Z)/Z$ has modulus 1 on the circumference. Since it vanishes somewhere other than $Z=0$, it is not a constant. Hence the modulus is less than 1 in $|Z|<1$, and the inequalities (2) and (3) [and so (4) and (5)] must hold.

The subsequent argument requires only the *existence* of a function with these properties. We can find its actual form [viz. (1)] by the following stages. We first represent the single sheeted unit circle of z on that of z', $z=r$ corresponding to $z'=0$; then the double z' circle, with winding point at the origin, on the simple z'' circle; finally the (simple) z'' circle on the (simple) Z circle, making $z''=0$ correspond to $Z=r$ (the last fulfils the requirement of making $z=0$, $Z=0$ correspond). The three transformations are:

$$\frac{z-r}{rz-1}=z',\ \sqrt{z'}=z'',\ z''=\frac{Z-\sqrt{r}}{\sqrt{r}Z-1}.$$

16 . 3. The more general transformation

$$\frac{Z-\sqrt{r}e^{i\phi}}{\sqrt{r}Z-e^{i\phi}}=\sqrt{\left(\frac{z-re^{i\phi}}{rz-e^{i\phi}}\right)} \tag{1}$$

also has the properties (4) and (5).

Let now $r_0e^{i\phi_0}$ be the point of $F(D)$ nearest 0. Then

$$\frac{\zeta_1-\sqrt{r_0}e^{i\phi_0}}{\sqrt{r_0}\zeta_1-e^{i\phi_0}}=\sqrt{\left(\frac{z-r_0e^{i\phi_0}}{r_0z-e^{i\phi_0}}\right)},\quad\text{or}\quad \zeta_1=f_1(z),$$

gives a ζ_1 regular and "schlicht" in D, and transforms D to D_1, say, interior to d. Since $f_1(z)$ is continuous in D' it transforms points of $F(D)$ into points of $F(D_1)=F_1$. Hence, if r_1 is the distance of 0 from F_1, we have $r_1>r_0$ (provided $r_0<1$). We now repeat the process on D_1, and so on, obtaining sequences (D_n), (r_n). It is in point of fact the case that for any D r_n tends necessarily to 1 (so that D_n tends to the unit circle), that $f_n\to f$ in D, and that (what the reader will then easily believe) $w=f(z)$ represents D on the w unit circle.

The actual argument is rather long and delicate, and we become a little awkwardly entangled in the structure of r_n and f_n. (We may ob-

serve, however, that the selection principle is not required.) There is, however, an alternative line of attack which, depending ultimately as before on the transformation (1), employs, so to say, a larger army of functions (in fact, the largest possible). Here there is much greater flexibility, and no irrelevant detail†.

Let D be any bounded domain. We may suppose $z=0$ an interior point. Consider the class of all functions $\phi(z)$ with the properties :

(i) ϕ is bounded in D,

(ii) ϕ is "schlicht" in D,

(iii) $\phi(0)=0$,

(iv) $\phi'(0)=1$.

We want to prove that some $\phi=\phi_0$ represents D on a circle. Now if a ϕ_0 exists (as it in fact does) the function ϕ_0 and the circular domain will have certain interesting minimal properties with respect to D. For example, $M(\phi)=M(\phi, D)$, the upper bound of $|\phi|$ for z of D, and $\mathfrak{A}(\phi)=\mathfrak{A}(\phi, D)$, the area of the transform of D by $w=\phi(z)$, both have their minimum values when ϕ is ϕ_0. This is indeed true if we allow ϕ to range over the class of functions subject to (iii) and (iv) only. Thus, let R be the radius of Δ_0 (the circle), let $Z(w)$ be the function inverse to $w=\phi_0(z)$, and let $\Phi(w)=\phi\{Z(w)\}$ (so that $\Phi(w)$ is w when ϕ is ϕ_0).

Then Φ, *qua* function of w in $|w|<R$, satisfies the conditions $\Phi(0)=0$, $\Phi'(0)=1$; we have

$$M(\phi, D)=M(\Phi, \Delta_0), \quad \mathfrak{A}(\phi, D)=\mathfrak{A}(\Phi, \Delta_0);$$

and it is enough to prove

$$M(\Phi, \Delta_0) \geqslant R, \quad \mathfrak{A}(\Phi, \Delta_0) \geqslant \pi R^2.$$

The first result is immediate, since (by Theorem 107)

$$R=R|\Phi'(0)| \leqslant RM\left(\frac{\Phi}{w}\right)=M(\Phi).$$

Next we have

(2) $$\mathfrak{A}(\Phi)=\iint |\Phi'(w)|^2 r\,dr\,d\phi,$$

where $w=re^{i\phi}$, the double integral is taken over Δ_0, and regions covered multiply are counted multiply in the area. For $|\Phi'(w)|^2$ is the Jacobian

† It would be possible to obtain something (but not enough) of this improvement by making the original process transfinite; the expanding sequence D'_n has a limit set D'_ω, we may repeat the process with D_ω in place of D, and so on transfinitely.

of the transformation from w to Φ†. Now for w of Δ_0 we have

$$\Phi'(w) = 1+2b_2 w+3b_3 w^2+\ldots,$$

and the right-hand side of (2) is (by Parseval's theorem)

$$2\pi \int_0^R (1+2^2 |b_2|^2 r^2+3^2 |b_3|^2 r^4+\ldots) r\,dr = \pi R^2+\pi \sum_{n=2}^{\infty} n |b_n|^2 R^{2n} \geqslant \pi R^2.$$

Suppose now that $N(\psi)$ is any number {such as $M(\psi)$ or $\mathfrak{A}(\psi)$} associated with a function ψ, and that a class of ψ, all defined in a fixed D, has a member ψ_0 for which N is a minimum. If ν is the lower bound of N (for varying ψ) there exists a sequence (ψ_n) for which $N(\psi_n) \to \nu$; we can, perhaps, select a subsequence of (ψ_n) converging in D to a limit-function, and this function may well turn out to be ψ_0. This suggestion we now follow up for the class of functions ϕ satisfying (i) to (iv). The argument can be carried through when N is either $\mathfrak{A}(\phi)$ or $M(\phi)$; the latter gives the simpler version.

16 . 4. After these preliminaries we make a fresh start. We take as known the properties (4) and (5) of the function (1) of § 16 . 1. They are established either by the argument of that sub-section or by the last fifteen lines of § 16 . 2; one of these counts as part of the official proof. What now remains is important but quite short

The class of functions ϕ satisfying (i) to (iv) exists, the function

$$\phi(z) = z$$

being a member of it. Let μ be the lower bound of $M(\phi)$ for all ϕ. Then there exists a sequence (ϕ_n) of functions ϕ for which $M(\phi_n) \to \mu$. ϕ_n is uniformly bounded in D for large n [since $|\phi_n| \leqslant M(\phi_n) < \mu+\epsilon$]. Hence, by Theorem 114, we can select a subsequence of the ϕ_n, with which we may now identify the original sequence, converging to a $\psi(z)$ in D. By Weierstrass's theorem (or by Theorem 113) $\psi'(0) = \lim \phi_n'(0) = 1$; and incidentally ψ is not a constant. By Lemma 8, Cor., ψ is "schlicht" in D. Further, $M(\psi) \leqslant \mu$. For if $M(\psi) > \mu' > \mu$ we have $|\psi(\zeta)| > \mu'$ for some ζ of D, $|\phi_n(\zeta)| > \mu'$ for large n, and so $M(\phi_n) \geqslant |\phi_n(\zeta)| > \mu'$ for large n, which is false. ψ now satisfies conditions (i) to (iv), and is a ϕ. Hence finally $M(\psi) \geqslant \mu$ and so $M(\psi) = \mu$.

I say now that the transform Δ of D by $w = \psi(z)$ is a circle, which establishes the theorem. If Δ were not a circle we could decrease its largest radius, which is μ, at the expense of the smallest. In fact, given

† Since we are concerned with the transform of the *whole* of Δ_0 a full discussion requires some limit-argument. This is given in most accounts of quadrature, and the point is in any case rather trivial. As we do not actually use the result we omit the details.

a z-domain Δ containing $z = 0$, interior to a circle d of radius μ and not identical with it, there exists a $\chi(z)$, with $|\chi'(0)| > 1$, which transforms Δ into a new domain interior to d; we have only to take $\chi(z) = \mu Z(z/\mu)$, where $Z(z)$ is the function (1) of § 16 . 3, and $re^{i\phi}$ is any point of $F(\Delta)$ interior to d. If now our Δ were not identical with $|w| < \mu$ we should have $|\chi\{\psi(z)\}| \leqslant \mu$. Then

$$\psi_1(z) = \frac{\chi\{\psi(z)\}}{\chi'(0)}$$

is "schlicht" in D, $\psi_1(0) = 0$, $\psi_1'(0) = \psi'(0) = 1$, and

$$M(\psi_1) \leqslant \frac{\mu}{|\chi'(0)|} < \mu.$$

This is impossible.

17. *The conformal representation of limit-domains in general.*

17 . 1. *The nucleus of a sequence of domains.*—Let (D_n) be a sequence of domains each containing $z = 0$. The nucleus N of the sequence is defined to be, either the greatest *domain* D, containing $z = 0$ and with the property $D'_- \subset D_n$ for any D_- and all large n; or, if no such domain exists, the single point $z = 0$. (In a nucleus there is a privileged point, taken to be the origin.)

We say further that D_n "converges into its nucleus" if every subsequence gives the same N.

Notes.—1. If *one* domain D has the "$D'_- \subset D_n$ $(n > n_0)$" property, so has ΣD, the summation being taken over *all* such D. A sum of domains *all containing* z_0 is a domain (z_1 and z_2 are connectible *via* z_0). Σ is therefore a domain, and the *greatest* one of its kind. N therefore exists (and is unique).

2. Let L be the set of points P such that P is contained in D_n for all large n; the set of interior points of L is a sum of domains, let M be the domain containing $z = 0$. N (supposed not the single point) is *not* necessarily the same as M. Suppose, *e.g.*, D_n is the unit-circle less the circle (taken closed) on the segment $\frac{1}{2} + \frac{1}{2}n^{-1}$ to $\frac{1}{2} + n^{-1}$ as diameter. M is the interior of the unit-circle, N is M less the point $z = \frac{1}{2}$. (Thus N and M need not even have the same connectivity.)

3. Examples. (i) D_n expanding and uniformly bounded. Here $N = \Sigma D_n$, and D_n converges into N. This is the most important case.

(ii) Contracting D_n. There exists a limit set L. If L, taken open,

is a *single* domain, then $N = L$. (L need not, however, be a single domain; see, for example, Fig. 7.)

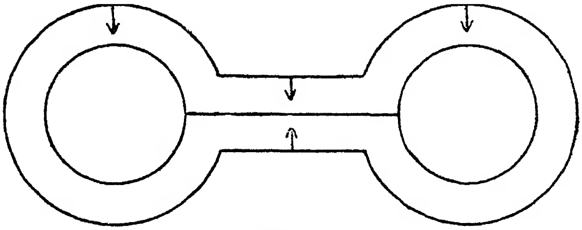

Fig. 7.

17.2. LEMMA 9.—*Given two bounded sequences of domains*† *(containing the origin), (D_n) and (Δ_n), let D and Δ be their nuclei, neither being a point. Let $\zeta = f_n(z)$ represent D_n on Δ_n, and let ϕ_n be the inverse function of f_n. Suppose now that*

$$f_n(0) = 0, \quad f_n \to f \text{ in } D, \quad \phi_n \to \phi \text{ in } \Delta.$$

Then $\zeta = f(z)$ represents D on Δ, and ϕ is the function inverse to f.

Observe that [since $\phi_n(0) = 0$] there is complete reciprocity: anything proved for D, f, ... applies also to Δ, ϕ, ...

The proof falls into four stages.

(*a*) *f and ϕ are not constant.* This is far from trivial. If f is constant its value is $f(0) = 0$. If $\zeta_0 \neq 0$ belongs to Δ, let $\mathrm{z} = \phi(\zeta_0)$, $\mathrm{z}_n = \phi_n(\zeta_0)$. Then $\mathrm{z}_n \to \mathrm{z}$. If now $\mathrm{z} \subset D$, there exists a D_- such that (i) z_n, $\mathrm{z} \subset D_-$ $(n > n_0)$, and so also (ii) z_n, $\mathrm{z} \subset D_n$ $(n > n_1 \geqslant n_0)$. From (ii) $f_n(\mathrm{z}_n) = \zeta_0$. From (i) $|f(\mathrm{z}_n) - f_n(\mathrm{z}_n)| \to 0$ as $n \to \infty$ (uniform convergence of f_n in D'_-). But the left side is $|0 - \zeta_0|$, and we have a contradiction. Therefore $\phi(\zeta_0)$ takes only values lying in $C(D)$ for $\zeta_0 \neq 0$ of Δ. Since $\phi(0) = \lim \phi_n(0) = \lim 0 = 0$ and ϕ (the uniform limit of ϕ_n) is continuous at $\zeta = 0$, this is impossible.

(*b*) *f is "schlicht" in D, ϕ "schlicht" in Δ.* This follows from (*a*) and Lemma 8.

(*c*) *The values of f (for z of D) lie in Δ, the values of ϕ in D.* By (*b*) $\zeta = f$ represents D on some domain δ. Given z of D, there exists a D_- containing 0 and z, and a D_0 such that $D'_- \subset D_0$ and $D'_0 \subset D$ (see Lemma 7). By $\zeta = f$ there correspond δ_- and δ_0 similarly related to δ and $\zeta = 0$. Now by the uniform convergence of f_n in D_0, $\zeta = f_n$ (n large) represents D_0 on some area differing arbitrarily little from δ_0,

† By a bounded sequence of domains we mean a sequence of uniformly bounded domains.

and therefore containing δ'_-. Since also $D_n \supset D_0$ (for large n) we see that $\Delta_n \supset \delta'_-$ for almost all n. *By the definition of* Δ, $\delta_- \subset \Delta$, and the value $f(z)$ belongs to Δ.

(*d*) $\subset \delta$. Let ζ_0 be any point of Δ; we have to prove it the f of some z. Now $z_n = \phi_n(\zeta_0) \rightarrow \phi(\zeta_0) = \mathrm{z}$, a point of D, by (*c*). Therefore there is a D_- such that z_n and z belong to D_- for large n. Hence

$$f(\mathrm{z}) - \zeta_0 = \{f(\mathrm{z}) - f(z_n)\} + \{f(z_n) - f_n(z_n)\}$$
$$\rightarrow 0 + 0 = 0,$$

by the continuity of f and the uniform convergence of f in D_- respectively. Hence $f(\mathrm{z}) = \zeta_0$: ζ_0 is a point of δ.

It follows now that $\zeta = f$ represents D on Δ. Finally, we have just seen that

$$f\{\phi(\zeta_0)\} = \zeta_0$$

for any ζ_0 of Δ; hence f and ϕ are inverses.

17.3. We prove next:

Given a bounded sequence of D_n *each containing* $z = 0$, *and that* D_n *converges into a nucleus* D, *not a point; also a bounded sequence* Δ_n, *with nucleus*† Δ. *Suppose now that* $\zeta = f_n(z)$ *represents* D_n *on* Δ_n, $f_n(0) = 0$, *and* $f_n \rightarrow f$ *in* D. *Then* (1) $\zeta = f(z)$ *represents* D *on* Δ, (2) Δ_n *converges into* Δ.

Case (i). Δ *not a point.*—Consider the inverses ϕ_n. We can select a subsequence giving $\Phi_m = \phi_{n_m} \rightarrow \Phi$, $F_m = f_{n_m} \rightarrow F = f$. Let Δ^* be the N of the $\Delta_{n_m} = \Delta^*_m$, D^* that of the $D^*_m = D_{n_m}$. Then $D^* = D$, $\Delta \subset \Delta^*$, and Δ^* is not a point. By Lemma 9, $\zeta = f$ represents D on Δ^*, and f, Φ are inverses. It follows now that $\phi_n \rightarrow \Phi$. If not, we can select different subsequences giving different limit-functions Φ and Φ_0, and nuclei (possibly the *same*) Δ^*, Δ^*_0. But since then $\zeta = f(z)$ represents D on Δ^* and on Δ^*_0, these last are identical. Then further $z = \Phi(\zeta)$, $z = \Phi_0(\zeta)$, defined in the same Δ^*, are both inverses of f, therefore identical. But if $\phi_n \rightarrow \Phi$ the subsequence originally chosen may be the whole sequence, and $\Delta^* = \Delta$. This proves both (1) and (2).

Case (ii). Δ *the point* $\zeta = 0$.—*We have to show* $f_n \rightarrow 0$ *in* D. If not, f is not constant, is therefore "schlicht" (Lemma 8), and therefore represents D on some domain δ containing $\zeta = 0$. Take δ_- containing $\zeta = 0$, and δ_0 such that $\delta'_- \subset \delta_0$, $\delta'_0 \subset \delta$, and let D_-, D_0 be the corresponding

† We slightly extend here the use of the symbol Δ; Δ may be a point.

z-domains (they contain $z=0$). $\zeta = f_n$ represents D_0 on something differing arbitrarily little from δ_0, therefore containing δ_-. It follows that $\delta_- \subset \Delta$, and this is impossible.

17.4. It is easy to prove a converse of the last theorem, viz.: *Let D_n, D, Δ_n be as before, and let $\zeta = f_n(z)$ represent D_n on Δ_n, $f_n(0)=0$, $f_n'(0)>0$. Then, given further that Δ_n converges into Δ, it follows that (for some f) $f_n \to f$ in D.*

If not, there exist two subsequences (f_{n_m}) with different limit-functions F_1 and F_2. With obvious notation we have $\Delta_1^* = \Delta_2^* = \Delta$ (by hypothesis). Therefore, by the direct result, $\zeta = F_1$ and $\zeta = F_2$ both represent D on Δ, with $F_1'(0) = \lim f_{n_m}'(0) \geqslant 0$ and similarly $F_2'(0) \geqslant 0$ This is impossible, by Theorem 120

Summing up we have:

THEOREM 122.—*Let (D_n), (Δ_n) be bounded sequences of domains, with nuclei D and Δ, where D is not a point, and let D_n converge into D. Let $\zeta = f_n(z)$ represent D_n on Δ_n, $f_n(0)=0$, $f_n'(0)>0$. Then the necessary and sufficient condition for f_n to converge to some limit-function f in D is that Δ_n should converge into Δ. If this happens $\zeta = f(z)$ represents D on Δ.*

It is an instructive exercise to write out the proof for the special case when D_n, Δ_n are expanding (and bounded uniformly), making the appropriate simplifications. The result, for this case, which we call (for reference) (A), is as follows:

If D_n, Δ_n are expanding and uniformly bounded, and tend to D and Δ, and if $f_n \to f$ in D, then $\zeta = f(z)$ represents D on Δ.

17.5. *Montel's proof of Theorem* 121.—D is a limit of expanding D_n which are *polygons*. The theorem, (B), that any (simply-connected) polygon can be represented on a circle, goes back to Schwarz (see, *e.g.*, Goursat's *Cours d'Analyse*, Vol. III). From (A) and (B) Theorem 121 follows, and this is Montel's proof. It is instructive to analyse further the proof of (A), especially so far as it depends on (C), Montel's theorem (Theorem 114). We want (C) only for a domain D_n *known* to be representable on a circle; it is therefore enough to prove (C) for a circle, a specially simple case.

The boundary problem.

18.1. We prove next a result of interest in itself, which we shall also have occasion to apply in the sequel.

THEOREM 123.—*Suppose that we are given a closed contour* Γ (*interior* Δ), *divided at* β *and* α *into* Γ_1 *and* Γ_2. *Suppose* (1) $\psi(\zeta)$ *is regular and bounded in* Δ, (2) ψ *is continuous in* Δ' *except possibly at* α, (3) $\psi \to a$ *as* $\zeta \to \alpha$ *along* Γ_1. *Then either* $\psi \to a$ *as* $\zeta \to \alpha$ *along* Γ_2, *in which case* $\psi \to a$ (*uniformly*) *as* $\zeta \to \alpha$ *in* Δ, *or else* ψ *does not tend to a limit as* $\zeta \to \alpha$ *along* Γ_2.

Suppose the result false, so that $\psi \to b \neq a$ along Γ_2. Let

$$\chi(\zeta) = \{\psi(\zeta) - a\}\{\psi(\zeta) - b\}.$$

Then $\chi \to 0$ along Γ_1 and Γ_2. We prove first that

$$\chi \to 0 \text{ uniformly as } \zeta \to \alpha \text{ in } \Delta.$$

By Lemma 5 we may suppose that α is the origin, that Γ_1, Γ_2 lie, except for α, to the right of the imaginary axis, and that $|\chi| \leqslant 1$ in Δ. [Our applications require only this special case, and are therefore independent of Lemma 5.] Let h be large and positive, and

$$\chi_1(\zeta) = \chi(\zeta)/(1 + h\zeta).$$

Given ϵ, choose $r(\epsilon)$ so that $|\chi_1| < \epsilon$ for points of Γ_1, Γ_2 in $|\zeta| \leqslant r$. Let $\delta(r)$ be a domain, with α on the boundary, cut off from D by $|\zeta| = r$. Now

$$|1/(1 + h\zeta)| \leqslant 1 \quad (\zeta \text{ in } \Delta'). \tag{1}$$

By choice of $h = h(\epsilon)$ we can make $|1/(1 + h\zeta)| < \epsilon$ for points of Γ outside $|\zeta| = r$. Hence $|\chi_1| \leqslant \epsilon$ for points of Γ other than α, whether in $|\zeta| \leqslant r$ or not, therefore (Theorem 102) also for Δ', and in particular in $\delta(r)$. That is, given ϵ, we have $|\chi| \leqslant \epsilon|1 + h(\epsilon)\zeta|$ for all ζ of $\delta(r)$. Hence $|\chi| < 2\epsilon$ for $|\zeta| < \text{Min}(r, 1/h)$, giving the result.

Return to ψ. Our hypothesis involves that in any $\delta(r)$ we can find a line λ such that for ζ of λ $\psi(\zeta)$ runs from a value near a to a value near b. The perpendicular bisector of (ab) meets the track of ψ, and the corresponding values of $\psi - a$, $\psi - b$ have moduli not less than $\frac{1}{2}|b - a|$. That is, there is a ζ in $\delta(r)$ for which $|\chi| \geqslant \frac{1}{4}|b - a|^2$. This contradicts $\chi \to 0$ and proves the theorem.

19. *The representation of* D' *on* Δ' *when* D *and* Δ *are bounded by curves.*

19.1. Let C be a simple closed contour in the z-plane, D its interior. Let Δ be the ζ-circle. By Theorem 121 there is an $f(z)$ such that

$\zeta = f(z)$ represents D on Δ. We show now that, with appropriate completion of its definition, $f(z)$ *is continuous in* D'. Given a point A, or $z = a$, of C we have to show that there exists an α such that $|f(z) - \alpha| < \epsilon$ in $d(A, r)$† $(r < r_0)$.

We observe first that, by Lemma 1 (§ 11.4), the lower bound of $|f|$ for z of d tends to 1 as $r \to 0$. We must now consider $\arg f$. Let $\sigma(A, r) = \sigma(r)$ be its oscillation in $d(A, r)$. We shall prove that $\sigma(r) \to 0$ with r. If not, let σ' be the upper limit of $\sigma(r)$, and $0 < \sigma_0 < \sigma'$, $\sigma_0 < \pi$. For some arbitrarily small r we have then $\sigma(r) > \sigma_0$. Considering always such r we can now draw in $d(r)$ a simple line l' on which $\operatorname{osc} \arg f > \sigma_0$. There corresponds in Δ a simple line λ' whose extremities P and Q' have arguments ϕ_1, ϕ_2' with $\phi_2' - \phi_1 > \sigma_0$. [The intuitive basis of the subsequent argument is as follows. λ' lies near the circumference. If it were an actual arc of the circumference we should have, roughly, $\phi(\zeta)$—*a small* on this arc, *bounded* on the rest of the circumference; therefore (compare, *e.g.*, § 9.8) small at points well in the interior of the circle, which is false.] Let $\phi_2 = \phi_1 + \sigma_0$; we then obtain a line PQ as in Fig. 8, with extremities of arguments ϕ_1, ϕ_2. Let M be the mid-point of RS, so that $OM = \cos \frac{1}{2}\sigma_0$.

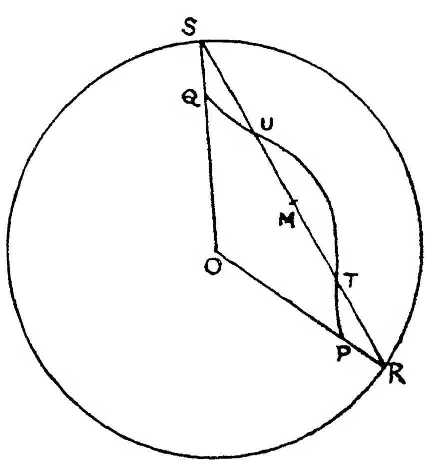

Fig. 8.

If r is small enough every point of PQ [corresponds to a point of $d(r)$ and] is distant more than OM from O. MR and MS therefore cross PQ, for

† We use the notation of the introduction, § 6.

the first time (from M) at, say, T and U. Let $\phi(\zeta)$ be the inverse function of f. By a transformation $\zeta = \zeta_1 e^{i\tau} + c$ we may suppose SR the real axis of ζ_1. Let $\phi(\zeta) - a = \phi_1(\zeta_1)$. The curve UT and the chord UT bound a domain Δ_1, and the reflection of Δ_1 in SR is evidently interior to the circle. Let $\Delta_2' = \Delta_1 + \bar{\Delta}_1 + \text{chord } (UT) + \text{boundaries}$. If ζ_1 lies in Δ_2' $\phi_1(\zeta_1)$, $\phi_1(\bar{\zeta}_1)$ are regular functions of their arguments, and their moduli do not exceed $\text{Max}\,|\phi(\zeta) - a| \leqslant K$, where K is the diameter of D. On the boundary *either* $|\phi_1(\zeta_1)|$ *or* $|\phi_1(\bar{\zeta}_1)| \leqslant m(r)$, since for ζ of $QUTP$ $\phi(\zeta)$ lies in $d(r)$, and so is at a distance not exceeding $m(r)$ from $z = a$. Hence

$$|\phi_1(\zeta_1)\,\phi_1(\bar{\zeta}_1)| \leqslant Km(r)$$

on the boundary of Δ_2', therefore also (Theorem 106) in Δ_2. In particular the inequality holds on the chord UT, where $\zeta_1 = \bar{\zeta}_1$, giving

$$|\phi_1(\zeta_1)| < \sqrt{\{Km(r)\}}.$$

[This is, in fact, a case of the result proved on p. 112 (fig. 5), but we have repeated the argument for completeness.] In particular

$$(1) \qquad |\phi(\zeta_M) - a| \leqslant \sqrt{\{Km(r)\}}.$$

The right-hand side tends to 0 as $r \to 0$. But ζ_M lies on $|\zeta| = \cos\frac{1}{2}\sigma_0$, and to this corresponds a z-contour distant $k(\sigma_0) > 0$ from the boundary C, contrary to (1). Hence $\sigma(r) \to 0$.

19.2. It follows that there exists an α satisfying $|\alpha| = 1$, such that $f(z) \to \alpha$ as $z \to a$ in D. This being true for every A f is evidently continuous in D'. It follows that $\zeta = f(z)$ makes correspond to every a of C a point α of the circumference $|\zeta| = 1$, and that α is a continuous function of a.

We show next that as a describes C, α moves continually in the same sense, and returns to its starting point when a does. If this were not so there would be two points a_1, a_2 of C to which the same α corresponded. To two non-intersecting simple lines L_1, L_2 from an interior point b of D to a_1, a_2 (see § 6) correspond in Δ two non-intersecting simple lines Γ_1, Γ from β to the same point α of $|\zeta| = 1$. Then $\phi \to a_1$ or a_2 as $\zeta \to \alpha$ along Γ_1 or Γ_2. Theorem 123 shows that this is impossible.

19.3. We prove finally that $\phi(\zeta)$ is continuous in Δ'. Let $q(r)$ be $B_r C_r$. To B_r, C_r correspond, in the sense of our work above, β_r and γ_r of

$|\zeta| = 1$, and to $q(r)$ corresponds a cross-cut $\beta_r \gamma_r$, or K_r, of Δ. $\delta(a, r)$ be the domain cut off by K_r which has a on its boundary. z near a give ζ near a (since f is continuous). Also to points of D separated, or not separated, by $q(r)$ correspond points of Δ separated, or not separated, by K_r, and conversely. Hence to ζ of $\delta(r)$ correspond z of $d(r)$, for which $|z-a| \leqslant m(r)$. Hence

$$|\phi(\zeta)-a| \leqslant m(r)$$

for ζ of $\delta(r)$. Since $m(r) \to 0$ with r, it follows that ϕ is continuous at boundary points, provided we define $\phi(a)$ to be a.

Summing up we have

THEOREM 124.—*If D is bounded by a simple closed contour the function f representing D on the unit circle Δ is continuous in D', while its inverse function ϕ is continuous in Δ'. The correspondence $\zeta = f(z)$ between D' and Δ' is one-one and bi-continuous.*

20.1. The more "regular" is the curve C bounding the domain D, the more "regularly" does the function $f(z)$ representing D on the unit ζ-circle behave at the boundary. Many propositions could be given corresponding to different degrees of regularity in hypothesis and conclusion. We shall content ourselves here with two.

We prove first the analogue of (part of) Theorem 123 for harmonic functions. We apply this in a moment, but it has a certain interest for its own sake.

THEOREM 125.—*Suppose that C is a closed simple Jordan curve and that D is its interior. Let the function u be harmonic and bounded in D, and continuous in D' except at a point a of C, and suppose that $u \to a$ as $z \to a$ along C_1 and C_2 (the two portions of C abutting at a). Then u is continuous also (in D') at $z = a$.*

A harmonic function of x, y remains harmonic if $x+iy$ undergoes a conformal transformation (see § 7.45). Hence we may suppose, in virtue of Theorem 124, that C is the unit circle, a is the point $z = 1$, and $a = 0$. We have then, for $r < 1$,

$$u(r, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(R, \psi)\, P\left(\frac{r}{R}, \psi-\theta\right) d\psi,$$

where $r < R < 1$.

By hypothesis we have

$$\lim_{R\to 1} u(R, \psi) = U(\psi) \quad (\psi \neq 0).$$

Let us *define* $U(0)=0$. Then in virtue of our hypotheses $U(\psi)$ is continuous in $(-\pi, \pi)$†. Since $u(R, \psi)\to U(\psi)$ p.p. and boundedly, we may take the limit $R\to 1$ under the integral sign. Hence u is the Poisson integral of the continuous U, and the theorem follows from Theorem 59.

20.2.—THEOREM 126.—*Let C be a closed simple Jordan curve, D its interior, and let $\zeta=f(z)$ represent D conformally on the unit ζ-circle Δ. Suppose now that C possesses a proper‡ tangent at $z=a$, and that l is a straight line through a. Then to any curve in D touching l at a corresponds a curve in Δ touching a straight line λ, depending only on l, through $\alpha=f(a)$. Further, the angle between two lines λ_1, λ_2 is the same as that between the corresponding l_1, l_2.*

Let $$u=\arg\frac{f(z)-\alpha}{z-a}=\arg\frac{\zeta-\alpha}{z-a}.$$

u is harmonic in D, and continuous in D' except at $z=a$, $\arg(\zeta-\alpha)$ is bounded, and the hypothesis of a tangent at $z=a$ implies that $\arg(z-a)$ is bounded also§. Finally, when z describes C, starting at $a+$ and ending at $a-$, $\arg(z-a)$ increases by π, and so does $\arg(\zeta-\alpha)$†. Hence u tends to the same limit whether $z\to a$ along C_1 or along C_2. Theorem 125 now shows that u is continuous at $z=a$, and the results of Theorem 126 follow without difficulty.

20.3. THEOREM 127.—*Suppose that the hypotheses of Theorem 126 are satisfied, and, in addition, that C has a continuous tangent in the*

† But it does not, of course, follow at once that u is the Poisson integral of U; this, indeed, is practically what we have to prove.

‡ z approaches a along the two branches of C in opposite directions (a cusp is excluded).

§ This is true for a *neighbourhood* of a, by the hypothesis. Moreover in the part of D for which $|z-a|>\delta$ $\arg(z-a)$ is continuous, and so bounded.

neighbourhood of $z = a$, or, more generally, that the chord from z to z' tends uniformly to the tangent at a as z and z' tend independently to $z = a$ on C. Then $\arg f'(z)$ *is continuous at $z = a$. Hence also: to every curve of continuous tangent issuing from a corresponds a similar curve issuing from a.*

This theorem is rather difficult (and the reader may ignore it if he wishes). Let $z = \phi(\zeta)$ be the function inverse to $\zeta = f(z)$, and let

$$\psi(\zeta, \tau) = \frac{\phi(\zeta e^{i\tau}) - \phi(\zeta)}{\zeta e^{i\tau} - \zeta} = \frac{z_1 - z}{\zeta_1 - \zeta}.$$

For fixed τ ψ is regular and never zero in Δ (note that $\zeta = 0$ requires special consideration), and continuous in Δ'. Hence

$$u(\zeta, \tau) = \arg \psi = \arg(z_1 - z) - \arg(\zeta_1 - \zeta)$$

is harmonic in Δ (in particular at the point $\zeta = 0$), and continuous, for fixed τ, in Δ'.

We now make the provisional assumption that there exists a G, independent of τ and ζ, such that

(1) $$|u| < G$$

upon $|\zeta| = 1$, and therefore in Δ'.

As $\tau \to 0$ we have $u(a, \tau) \to \omega$, where ω is the angle between the tangent to C at a and the tangent to $|\zeta| = 1$ at a. Now the hypothesis about the chord zz' is equivalent (in virtue of the continuity of f and ϕ) to the following assertion: For a given ϵ there exist η, δ such that, on an arc of $|\zeta| = 1$ of length η on each side of $a = e^{i\psi_0}$,

(2) $$|u(\zeta, \tau) - \omega| < \epsilon \quad (|\tau| < \delta).$$

We proceed to prove that a similar inequality holds for ζ near a and *interior* to Δ. Let $U^*(\psi) = U^*(\psi, \tau)$ be the boundary function of the continuous function $u - \omega$ at $e^{i\psi}$, so that $u - \omega$ is the Poisson integral of U^*. Now U^* is bounded above (for varying τ as well as varying ψ) by $G + |\omega|$ and is not greater than ϵ for points of the arc $(\psi_0 - \eta$,

† The two contours are described in the same sense (Theorems 124, 117).

Hence, if $\zeta = re^{i\theta}$, $|\tau| < \delta$, we have

$$2\pi\{u(\zeta, \tau) - \omega\} = \int_{\psi_0-\eta}^{\psi_0+\eta} U^*(\psi)\, P(r, \psi-\theta)\, d\psi + \left(\int_{-\pi}^{\pi} - \int_{\psi_0-\eta}^{\psi_0+\eta}\right) U^*\, P d\psi$$

$$= \int_{(1)} + \int_{(2)},$$

$$2\pi\,|u-\omega| \leqslant \epsilon \int_{(1)} P d\psi + (G + |\omega|) \int_{(2)} P(r, \psi-\theta)\, d\psi$$

$$\leqslant 2\pi\epsilon + (G + |\omega|)\, h\,(\eta, r, \theta),$$

where h is independent of τ, and tends uniformly to zero as $\zeta = re^{i\theta} \to e^{i\psi}$ (Theorem 59). We have proved, therefore, that there exist η_1, δ_1 such that

(3) $$|u-\omega| < \epsilon \quad (|\zeta - a| < \eta_1,\ |\tau| < \delta_1).$$

If now we call Δ_1 the part of Δ for which $|\zeta - a| < \eta_1$, then, for any (interior) ζ of Δ_1, we have, on making $\tau \to 0$ and so $u \to \arg \phi'(\zeta)$,

$$|\arg \phi'(\zeta) - \omega| \leqslant \epsilon.$$

It follows that $\arg \phi'(\zeta)$ is continuous at the point a. Our result follows, since $f'(z) = 1/\phi'(\zeta)$.

20.4. It remains to remove the assumption (1). Consider the domain $d(a, r)$ and the corresponding $\delta(a, r)$ of Δ. δ is bounded by an arc $\beta\gamma$ of the circumference of $|\zeta| = 1$ and a simple line from β to γ in Δ'. The main ideas of our argument are: (i) The hypothesis of the theorem implies the result (1) for the domain d provided r is small enough, (ii) it is plausible that only the neighbourhood of $z = a$ is really relevant to the problem.

Let $\zeta_1 = f_1(\zeta)$ represent $\delta(a, r)$ on $|\zeta_1| < 1$, or Δ_1; then, by Theorem 124, the correspondence extends to the boundaries, and

$$\zeta_1 = f_1\{f(z)\} = F(z)$$

(say) is continuous in d'. Now we have, by hypothesis, if r is small enough,

(1) $$|\arg(z' - z)| < K$$

for z, z' of the part of the boundary of d belonging to C. An inequality of the type of (1) holds also if one of z, z' belongs to C and the other to the circular arc $\beta\gamma$, while yet another holds if both belong to $\beta\gamma$. Also,

of course,

$$|\arg(\zeta' - \zeta)| \leqslant 2\pi\dagger.$$

It follows that the condition (1) of § 20.3 is satisfied for the function $F(z)$ that represents d on Δ_1, and we conclude that $\arg F'(z)$ is continuous at $z = a$. But $f_1(\zeta)$ makes part of a circular arc (containing the point $\zeta = \alpha$) correspond to part of a circular arc. It follows, by Theorem 118, that f_1 is regular at internal points of the arc; and, further, that the conformal representation effected by f_1 extends to a domain on the other side of the arc, in virtue of which fact $f_1'(\zeta)$ does not vanish at any internal point of the arc. Consequently $\arg f_1'(\zeta)$ is continuous in $|\zeta| \leqslant 1$ at $\zeta = \alpha$. Hence, since $f(z)$ is continuous in D', $\arg f_1'\{f(z)\}$ is continuous at $z = a$; and hence, finally,

$$\arg f'(z) = \arg F'(z) - \arg f_1'(\zeta)$$

is continuous at $z = a$.

† It is supposed that z, z' (or ζ, ζ') start from some fixed pair of positions, and vary in any way subject to the restrictions that they do not cross each other and that neither crosses $z = a$.

CHAPTER II.

The present chapter deals with the following problem. Suppose that a function $f(z)$, regular in the unit circle, is restricted by the condition of never taking a certain value, or set of values. What influence has this condition on the behaviour of the function; in particular, how does it restrict the maximum modulus $M(\rho, f)$ [or the means $M_\lambda(\rho, f)$] and the n-th coefficient c_n? The most striking result in this field is the famous theorem of Picard concerning functions that have 0 and 1 as missing values. We consider the problem of missing values in a generalized form, in which we suppose that the point $w = f(z)$ moves, for varying z, on some given Riemann surface. Our problem leads naturally to the discussion of the class of functions $f(z)$ that are "schlicht" in the unit circle of z: this is in any case a subject of great intrinsic interest, and we devote a special section to it.

We begin with a section on "subharmonic" functions. The subject is not *prima facie* connected with our main problem, but some of our arguments find here their most fundamental application.

21. *Subharmonic functions.*

21.1. A real function $w(x, y)$, or for brevity $w(z)$, where $z = x+iy$, is called subharmonic in a domain D if it satisfies a certain pair of conditions (A) and (B). Condition (A) is of the nature of a restriction to continuity. It is essential to have *some* such restriction. We assume, in the first place, that for z of D

(A) (1) $$w(z) \geqslant \overline{\lim_{\zeta \to z,\, \zeta \neq z}}\, w(\zeta);$$

and we assume further that

(A) (2) $$w(z) < \infty \quad \text{in } D.$$

The value $-\infty$ for w is permitted.

The second condition (B) may take a number of forms $(B_1), \ldots, (B_4)$, all of which, however, we shall in the end find to be equivalent.

$(B_1), \ldots, (B_4)$ require respectively that for every (x, y) of D we shall have

$$(B_1) \qquad w(x, y) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} w(x+r\cos\theta,\ y+r\sin\theta)\,d\theta$$

for all small r;

$$(B_2) \qquad w(x, y) \leqslant \frac{1}{\pi r^2}\int_{-\pi}^{\pi} d\theta \int_0^r w(x+\rho\cos\theta,\ y+\rho\sin\theta)\,\rho\,d\rho$$

for all small r;

$$(B_3) \qquad w(x, y) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} w(x+r\cos\theta,\ y+r\sin\theta)\,d$$

for some (arbitrarily) small r;

$$(B_4) \qquad w(x, y) \leqslant \frac{1}{\pi r^2}\int_{-\pi}^{\pi} d\theta \int_0^r w(x+\rho\cos\theta,\ y+\rho\sin\theta)\,\rho\,d\rho$$

for some (arbitrarily) small r.

It follows from any form of (B) that for z of D $w(z) \leqslant \overline{\lim}_{\zeta\to z} w(\zeta)$. Combining this with (A) we see that *a function* $w(z)$ *subharmonic in* D *is upper-semi-continuous in* D, *that is, it satisfies*

$$(A^*) \qquad w(z) = \overline{\lim_{\zeta\to z}}\, w(\zeta)$$

at every point z *of* D.

We continue, however, to distinguish "condition (A*)" from "conditions (A)", that is to say, conditions (A) (1) and (2), but not necessarily (A*).

21.2. A function $w(z)$ satisfying conditions (A) in a bounded closed set E is bounded above in E (by the usual covering argument for the continuous case). Hence $w(z)$ is bounded above in any bounded closed subset E of D. It attains its upper bound in any such set. For if G, the upper bound, were unattained, the function $w^* = (G-w)^{-1}$ would satisfy conditions (A) in E, but would not be bounded above in E. Obviously $w(z)$ is measurable, and the various integrals in the conditions (B) exist, since $w(z)$ is bounded above on the circles concerned. These integrals may, however, have the value $-\infty$.

A function $w(z)$, *satisfying conditions* (A), *is the limit-function of a decreasing sequence* $\{w_n(z)\}$ *of continuous functions in any bounded closed subset* E *of* D.

We may suppose that $w(z)$ is not identically $-\infty$ [when we may take $w_n(z) = -n$]. Let ζ be any point of the (x, y)-plane, and let us write, for any fixed integer $n \geqslant 1$,

$$w_n(\zeta) = \operatorname*{upper\ bound}_{z \text{ in } E} \{w(z) - n\delta(z, \zeta)\}, \tag{1}$$

where $\delta(z, \zeta)$ denotes the distance between z and ζ: $w_n(\zeta)$ is finite, and there exists a point $z_0 = z_0(\zeta)$ in E such that

$$w_n(\zeta) = w(z_0) - n\delta(z_0, \zeta).$$

For any other point ζ_1

$$\begin{aligned} w_n(\zeta_1) &\geqslant w(z_0) - n\delta(z_0, \zeta_1) \geqslant w(z_0) - n\{\delta(z_0, \zeta) + \delta(\zeta, \zeta_1)\} \\ &= w_n(\zeta) - n\delta(\zeta, \zeta_1). \end{aligned}$$

Hence $w_n(\zeta_1) \geqslant w_n(\zeta) - \epsilon$, if $\delta(\zeta, \zeta_1) \leqslant \epsilon/n$. Since we may interchange ζ and ζ_1, $w_n(\zeta)$ is continuous (in the whole plane). Also $w_n(\zeta) \geqslant w_{n+1}(\zeta)$, and if we choose ζ as a z_1 in E, (1) gives $w_n(z_1) \geqslant w(z_1)$. Suppose first that $w(z_1) > -\infty$, and let z' be another point of E. By (A) (1),

$$w(z') \leqslant w(z_1) + \epsilon; \quad w(z') - n\delta(z', z_1) \leqslant w(z_1) + \epsilon,$$

provided that $\delta(z', z_1) < \delta(\epsilon)$. On the other hand, $w(z)$ is bounded above in E. Hence there is a G such that

$$w(z') - n\delta(z', z_1) \leqslant G - n\delta$$

if $\delta(z', z_1) \geqslant \delta$. The right-hand side does not exceed $w(z_1) + \epsilon$ if n is large enough. For such n, therefore, $w(z') - n\delta(z', z_1) \leqslant w(z_1) + \epsilon$ for *all* z' of E, *i.e.* $w_n(z_1) \leqslant w(z_1) + \epsilon$. This proves that $w_n(z_1) \to w(z_1)$. The proof is similar if $w(z_1) = -\infty$.

21.3. The inequality in (B_1) and (B_3) asserts that $w(x, y)$ does not exceed the average of w over the circumference of the circle of radius r round (x, y); the inequality in (B_2) and (B_4) asserts that $w(x, y)$ does not exceed the average of w over the area of the same circle†. For a *harmonic* func-

† Our results can be extended to higher averaging processes; for example, to the inequality

$$w(x, y) \leqslant \frac{1}{r}\int_0^r dr \frac{1}{r}\int_0^r dr \ldots A(r)$$

or *all* (*or some*) *small* r, where $A(r)$ is one of the averages considered in the text.

tion u, of course, $u(z)$ is equal to each of the averages; also u and $-u$ are both subharmonic. If $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$, $\mathbf{W}_4$ are the classes of functions corresponding to the four definitions, it is trivial that $\mathbf{W}_2$ contains $\mathbf{W}_1$, that $\mathbf{W}_3$ contains $\mathbf{W}_1$, and that $\mathbf{W}_4$ contains $\mathbf{W}_2$ and so $\mathbf{W}_1$. The rest of the proof of the equivalence of the four classes depends on Theorem 201 below.

Consider the special class of w with continuous second derivatives in D. Let the circle d, $|z-z_0|\leqslant r$, be contained in D, and let

$$J(r)=\frac{1}{2\pi}\int_{-\pi}^{\pi} w(z_0+re^{\theta i})\,d\theta.$$

We have, as a case of (3) § 7 . 12, the formula

$$\frac{dJ}{dr}=-\frac{1}{2\pi r}\int\frac{\partial w}{\partial n}\,ds=\frac{1}{2\pi r}\iint_d \Delta w\,dx\,dy. \tag{1}$$

If $\Delta w\geqslant 0$ in D this gives $J(r)\geqslant J(0)=w(z_0)$ for every z_0 of D and every small r, so that w is subharmonic by every definition. Conversely, the condition $\Delta w\geqslant 0$ in D is necessary for a w (of the special class) to be subharmonic in D by *any* definition, whence the equivalence of all definitions within the class. If the condition is satisfied, then $J(r)$ is, by (1), monotonic increasing (in the wide sense) in r (so long as d is contained in D).

If all subharmonic w belonged to the special class, many arguments would be as straightforward as this one. The special class is valuable in suggesting results, and the "$\Delta w\geqslant 0$" tendency should always be borne in mind. [It is, in fact, true, though we must not prove it now, that for a w subharmonic in D, Δw exists in a generalized sense, and is not less than 0, p.p. in D.] But quite practical applications force us to consider the wider class, and we have to find other arguments, no longer, but more delicate.

21 . 4. A function u will be called a harmonic frontier majorant of a function g for the domain D if u is harmonic in D and continuous in D', and $u(z)\geqslant\overline{\lim}\,g(\zeta)$ ($\zeta\to z$ in D) for points z of $F(D)$, the frontier of D.

THEOREM 201. *Suppose that w is subharmonic, by any one of the definitions, in the bounded domain D, and that u is a harmonic frontier majorant of w for D. Then $w\leqslant u$ for all points of D.*

Conversely, a function w, satisfying conditions (A) *in D, is subharmonic in D according to the definition* (B_1) *provided that $w\leqslant u$ in D_1 for every domain D_1 and function u which are such that D_1' is contained in D and u is a harmonic frontier majorant of w for D_1.*

COROLLARY 1.—*The four definitions are equivalent.*

COROLLARY 2. —*A function w, subhbarmonic in a bounded D and satisfying*

$$\overline{\lim_{\to z}}\, w(\zeta) \leqslant G$$

at every point z of $F(D)$, satisfies $w \leqslant G$ in D.

COROLLARY 3. *Let (B_i^*) denote what the inequality (B_i) becomes when the sign $\leqslant$ is replaced by $=$. Then a continuous function w, satisfying any one of the four conditions (B_i^*) in a bounded D, is necessarily harmonic in D.*

Suppose the first part of the theorem false. Define

$$d(z) = \overline{\lim_{\zeta \to z}} \{w(\zeta)\} - u(z)$$

for z of D' and ζ in D. This is $w(z)-u(z)$ for (interior) z of D, and somewhere takes positive values. Also† $d(z)$ satisfies conditions (A) in D'; it therefore attains an absolute maximum $M > 0$ at a point P of D', and the set, E say, of such points is closed. Let Q, or (x_0, y_0), be a point of $F(E)$, so that $d(Q) = M$, and [since $d \leqslant 0 < M$ at points of $F(D)$], Q is an interior point of D. Consider now the circumference C_r of small radius r round Q. This must contain a point R belonging to CE, at which therefore $m = d(R) < M$. Then, by the upper-semi-continuity, $d < m+\delta < M$ for points of C_r in a certain interval of θ, whence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d(x_0+r\cos\theta,\ y_0+r\sin\theta)\,d\theta < M = d(x_0, y_0);$$

and so, since the average of the harmonic function u is equal to $u(x_0, y_0)$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} w(x_0+r\cos\theta,\ y_0+r\sin\theta)\,d\theta < w(x_0, y_0).$$

This is true for *all* small r, which gives a contradiction if our definition is either (B_1) or (B_3). Finally, the area of the interior of C_r contains a point of CE (for example, an R associated with $C_{\frac{1}{2}r}$), and a precisely similar argument, for averages over the area instead of over the circumference, disposes of the remaining cases (B_2) and (B_4). The first part of the theorem is therefore established.

† Any upper limit $\overline{\lim}\, F(\zeta)$ clearly satisfies (A) (1).

Consider now the converse. Given a point (x_0, y_0) of D, consider a circumference C_r of radius r surrounding it (containing only points of D). On C_r, w is the limit-function of a decreasing sequence of continuous functions w_n (§ 21.2). Let u_n be the function harmonic in the interior of C_r and agreeing in value with w_n on C_r. This is a harmonic frontier majorant of w_n, and so of w, on C_r. By hypothesis $w(x_0, y_0) \leqslant u_n(x_0, y_0)$, or

$$(1) \qquad w(x_0, y_0) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} w_n(x_0+r\cos\theta,\ y_0+r\sin\theta)\,d\theta.$$

Since the interchange $\lim \int w_n = \int \lim w_n$ is permissible for a monotonic sequence [Theorem 11, Cor.], we have, by taking limits in (1),

$$w(x_0, y_0) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} w(x_0+r\cos\theta,\ y_0+r\sin\theta)\,d\theta,$$

and, since this is true for all small r, w satisfies the definition (B_1). This proves the converse.

Next, the complete theorem shows that a function subharmonic according to any definition is subharmonic according to definition (B_1); conversely it is trivial (as was pointed out in § 21.3) that a function subharmonic according to (B_1) is subharmonic according to any of the other definitions. This proves Corollary 1.

In Corollary 2 (a generalization of the maximum principle for harmonic functions) G is a harmonic majorant.

Finally, in Corollary 3, both w and $-w$ are subharmonic; inside any circle C (in D) w is consequently both not greater and not less than the harmonic function agreeing with w on the circumference; w is harmonic inside C.

THEOREM 202.—*If Δ is a conformal transformation of D, then a function subharmonic in D transforms into a function subharmonic in Δ.*

The converse part of Theorem 201 provides, in fact, an alternative definition of a subharmonic function in which the conditions are invariant under conformal transformation.

This proof is curiously indirect. An easy direct one would be available by means of

$$\Delta_{\xi,\eta} w = \left|\frac{dz}{d\zeta}\right|^2 \Delta_{x,y} w \geqslant 0,$$

if we might assume continuity of the second derivatives.

THEOREM 203. — *Let w be subharmonic in a domain containing a circle $|z-z_0| \leqslant r$, and let u be the Poisson integral of the values of w on the circumference. Then $w \leqslant u$ for interior points of the circle.*

N.B.—It must be borne in mind that $\int w(re^{i\theta})\,d\theta$ may be $-\infty$; in this case the Poisson integral u is evidently $-\infty$ at all interior points of the circle, and the theorem asserts that then the same is true of w.

Let $\zeta_0 = z_0 + \rho_0 e^{i\theta_0}$ be an interior point of the circle. A linear transformation of $|z-z_0| \leqslant r$ into itself which transforms ζ_0 into the centre transforms w into a subharmonic w^* (Theorem 202). Also it transforms the Poisson integral $u(\zeta_0)$ into

$$u^*(z_0) = \frac{1}{2\pi}\int w^*(z_0 + re^{i\theta})\,d\theta$$

[see § 7 . 45 (ii)]. Thus

$$u(\zeta_0) = \frac{1}{2\pi}\int w^*(z_0 + re^{i\theta})\,d\theta \geqslant w^*(z_0) = w(\zeta_0).\dagger$$

THEOREM 204.—*Suppose that w is subharmonic in $|z-z_0| < R$, and let $u(z, r)$ be the Poisson integral of the values of w on the circumference C_r of centre z_0 and radius $r < R$. Then, for fixed z, u is an increasing function of r (for $r > |z-z_0|$). In particular $\int w(z_0 + re^{i\theta})\,d\theta$ increases with r.*

If $r' > r$ we have $w(\zeta) \leqslant u(\zeta, r')$ for ζ of C_r (Theorem 203). Hence

$$\begin{aligned} u(z, r) &= \{\text{Poisson integral of } w(\zeta)\} \\ &\leqslant \{\text{Poisson integral of } u(\zeta, r')\} = u(z, r'). \end{aligned}$$

COROLLARY. *If w is not everywhere $-\infty$ in D then the integral of w over any area D'_- is finite (not $-\infty$).*

In any bounded D'_- w is bounded above, and so effectively of constant (negative) sign. We observe now that the integral of w over either the circumference or the area of a circle contained in D is finite. For by the present theorem it is enough to prove this for the circumference; if the circumference integral is $-\infty$ then $w = -\infty$ in the interior (Theorem 203); the integral of w along the circumference of any circle contained in D and intersecting the first is $-\infty$; and by a chain of circles $w = -\infty$

† If w is continuous on the circumference u is a harmonic majorant of w (Theorem 60), and the result of Theorem 203 follows from Theorem 201. This line of argument could be extended with a little trouble to cover the discontinuous case also.

at any assigned point of D, contrary to hypothesis. Finally, if the integral of w over some area D'_- is $-\infty$ it will be $-\infty$ also for some area of arbitrarily small diameter, and so for a circular area (containing this) contained in D, which we have seen to be impossible.

21.5. THEOREM 205.—*A finite sum, or a uniformly convergent infinite sum*† *of functions subharmonic in D is subharmonic in D. The function* $\text{Max}\,(w_1, w_2, \ldots, w_n)$, *where* $w_1, w_2, \ldots, w_n$ *are subharmonic in D, is subharmonic in D. If* $k \geqslant 1$ *and w is a non-negative function subharmonic in D, then* w^k *is subharmonic in D.*

For the last result we have, in respect of condition (B),

$$w^k(z_0) \leqslant \left(\frac{1}{2\pi}\int w(z_0+re^{i\theta})\,d\theta\right)^k \leqslant \frac{1}{2\pi}\int w^k(z_0+re^{i\theta})\,d\theta$$

(Theorem 1). The other cases of condition (B), and all cases of conditions (A), may be verified immediately.

THEOREM 206.—*Let* $\lambda > 0$, $k \geqslant 1$. *If* $f(z)$ *is regular at every point of D and* $|f|$ *is one-valued in D, then* $|f|^\lambda$, $\log|f|$, *and*

$$\overset{+}{\log}|f| = \text{Max}\,(0, \log|f|)$$

are subharmonic in D. Further, $|z|^\mu |f|^\lambda$ *is subharmonic, for every real* μ, *in the domain consisting of D less the point* $z = 0$. *If u (not necessarily real) is harmonic in D, then* $|u|^k$ *is subharmonic in D.*

For $|f|^\lambda$ we have to show that for a z_0 of D

$$|f(z_0)|^\lambda \leqslant \frac{1}{2\pi}\int |f(z_0+re^{i\theta})|^\lambda\,d\theta$$

for sufficiently small r. This is evidently true if $f(z_0) = 0$ If $f(z_0) \neq 0$ we have $f \neq 0$, and so f^λ regular, in some circle $|z-z_0| \leqslant r$. Then

$$|f^\lambda(z_0)| = \left|\frac{1}{2\pi}\int f^\lambda(z_0+re^{i\theta})\,d\theta\right| \leqslant \frac{1}{2\pi}\int |f|^\lambda\,d\theta.$$

A similar proof applies to $|z|^\mu|f|^\lambda$ and to $\log|f|$ [harmonic where $f(z_0) \neq 0$]. $\overset{+}{\log}|f|$ is the maximum of two subharmonic functions, therefore subharmonic.

† If this is interpreted as $\Sigma c_n f_n$ (coefficients c_n other than *unity*) the coefficients are to be *positive*.

Finally, if u is harmonic,

$$|u(z_0)|^k = \left|\frac{1}{2\pi}\int u(z_0+re^{i\theta})\,d\theta\right|^k \leqslant \frac{1}{2\pi}\int |u(z_0+re^{i\theta})|^k\,d\theta,$$

so that $|u|^k$ is subharmonic.

21.6. THEOREM 207.—(i) *Suppose that w is subharmonic in $r_1<|z|<r_2$ and not everywhere $-\infty$. Then*

$$I(w, \rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} w(\rho e^{i\theta})\,d\theta$$

is a continuous convex function of $\log\rho$ in $r_1<\rho<r_2$.

(ii) *Suppose that w is subharmonic in $|z|<r$ and not everywhere $-\infty$. Then $I(w, \rho)$ is a continuous increasing function of ρ in $0\leqslant\rho<r$, continuity at $\rho=+0$ being interpreted as $\lim_{\rho\to 0} I(w, \rho) = w(0) = -\infty$ in case $w(0)=-\infty$.*

(i) Let $r_1<\rho_1<\rho<\rho_2<r_2$, and let

$$\log\rho = t\log\rho_2+(1-t)\log\rho_1 \quad (0<t<1).$$

There exists a decreasing sequence of continuous functions $w_p(z)$ decreasing to $w(z)$ for all z in $\rho_1\leqslant|z|\leqslant\rho_2$ [§ 21.2]. By Theorem 71† there exists a function

$$(1)\quad u = u_p = k\log\rho+\sum_0^\infty(a_n\cos n\theta+b_n\sin n\theta)\rho^n + \sum_1^\infty(c_n\cos n\theta+d_n\sin n\theta)\rho^{-n},$$

continuous in $\rho_1\leqslant\rho\leqslant\rho_2$, harmonic in $\rho_1<\rho<\rho_2$, and agreeing in value with w_p on the boundaries of this annulus. u is a harmonic frontier majorant of w for the annulus; hence

$$(2)\quad I(w, \rho)\leqslant I(u, \rho) = k\log\rho+a_0 = t(k\log\rho_2+a_0)+(1-t)(k\log\rho_1+a_0)$$
$$= tI(u, \rho_2)+(1-t)I(u, \rho_1) = tI(w_p, \rho_2)+(1-t)I(w_p, \rho_1).$$

Since w_p decreases to the limit w, $I(w_p)\to I(w)$, and (2) gives in the limit

$$(3)\qquad I(w, \rho)\leqslant tI(w, \rho_2)+(1-t)I(w, \rho_1).$$

This proves the convexity.

Next, if $I(w, \rho)$ is $-\infty$ for any ρ of (ρ_1, ρ_2), it must, by the convexity, be $-\infty$ for an interval of values of ρ; this, however, would involve

† This is the sole, but essential, occasion on which we appeal to this theorem.

the integral of w being $-\infty$ over the area of an annulus, contrary to Theorem 204, Corollary. Thus $I(w, \rho)$ is finite in the open interval (r_1, r_2).

Finally, w is bounded above in $\rho_1 \leqslant |z| \leqslant \rho_1$ [§ 21.2]; so therefore is $I(w, \rho)$, and being convex it must be continuous. This completes the proof of (i).

In (ii) we know already that $I(w, \rho)$ is increasing in $0 \leqslant \rho < r$ [Theorem 204], and continuous in $0 < \rho < r$. For the remaining result, continuity at $\rho = +0$, we need only observe that on the one hand

$$I(w, 0) = w(0) \leqslant I(w, \rho)$$

for small ρ, and on the other $w(z)$ and so† $I(w, \rho)$ is less than $w(0)+\epsilon$ for $\rho < \delta(\epsilon)$ [by (A) (1)], whence $\lim_{\to 0} I(w, \rho) = w(0) = I(w, 0)$.

21.7. Theorem 207 gives, in particular, interesting results about the mean values of analytic functions and harmonic functions. We recall the definition

$$M_\lambda(\rho, f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\rho e^{i\theta})|^\lambda \, d\theta\right)^{1/\lambda} \quad (\lambda > 0).$$

THEOREM 208.—*Suppose that $f(z)$ is regular in $|z| < r$. Then $M_\lambda(\rho, f)$ is a continuous increasing function of ρ, and $\log M_\lambda$ is a continuous convex function of $\log \rho$, in $0 < \rho < r$. The last result holds further in $r_1 < \rho < r_2$ if f is regular in the annulus only (or if f is regular at each point of the annulus and $|f|$ is one valued).*

Similar results hold for harmonic functions u provided that $\lambda \geqslant 1$.

The first part follows from Theorem 207, since $|f|^\lambda$ is subharmonic (Theorem 206).

In the second, it is enough to prove the result in an arbitrary smaller closed annulus; we may suppose $r_1 > 0$, and that $w = |z|^\mu |f|^\lambda$ is continuous (and subharmonic) in $r_1 \leqslant \rho \leqslant r_2$. Following the plan of the proof of Theorem 108 we choose μ so that

$$r_1^\mu M_\lambda^\lambda(r_1, f) = r_2^\mu M_\lambda^\lambda(r_2, f) = T. \tag{1}$$

By Theorem 207, $I(w, \rho)$ is a continuous convex function of $\log \rho$. Since the extreme values (at $\rho = r_1, r_2$) are equal to T we have

$$I(w, \rho) \leqslant T = \{I(w, r_1)\}^t \{I(w, r_2)\}^{1-t}$$

† Supposing $w(0)$ is finite: the case $w(0) = -\infty$ needs only obvious modifications.

for every t; and choosing $t = t(\rho)$ to satisfy

(2) $$\rho = r_1^t r_2^{1-t}$$

we have

$$\rho^\mu M_\lambda^\lambda(\rho) \leqslant \{r_1^\mu M_\lambda^\lambda(r_1)\}^t \{r_2^\mu M_\lambda^\lambda(r_2)\}^{1-t}$$

$$= \rho^\mu M_\lambda^{\lambda t}(r_1) M_\lambda^{\lambda(1-t)}(r_2),$$

$$\log M_\lambda(\rho) \leqslant t \log M_\lambda(r_1) + (1-t) \log M_\lambda(r_2),$$

which expresses the required convexity.

The results for harmonic functions require only obvious modifications in the argument.

21.8. The following result, though we do not need it in the sequel, is so powerful and striking, and so easily and attractively proved, that we include it to end the present section†.

A function w, subharmonic in D, is the limit function in D of a decreasing sequence of continuous subharmonic *functions w_n; more generally, is the limit function of a decreasing sequence of subharmonic w_n with continuous partial derivatives of any assigned order.*

We may suppose w is not always $-\infty$ (when we can take $w_n = -n$).

Let us define $w_n(z)$ to be the average, $\mathcal{A}_n w$ say, of w over the area of the circle of radius $1/n$ about z‡. By Theorem 207 (ii) $w_n \to w$ decreasingly at each z of D. Since w is Lebesgue-integrable [Theorem 204, Corollary], w_n is continuous (in any D'_- for large enough n)§. Finally, if we denote by $\mathcal{A}_R$ an average over a circumference of radius R, we have relations which we can write briefly and intelligibly as

$$\mathcal{A}_R(w_n) = \mathcal{A}_R(\mathcal{A}_n w) = \mathcal{A}_n(\mathcal{A}_R w) \geqslant \mathcal{A}_n(w) = w_n.$$

These show that w_n satisfies condition (B).

This proves the result about continuous w_n. We can repeat the averaging process, obtaining $\mathcal{A}_n(w_n)$, $\mathcal{A}_n\{\mathcal{A}_n(w_n)\}$, ... It is hardly necessary to give a formal proof that a high enough average leads to a function with continuous partial derivatives of any assigned order

† For a report on the state of the theory of subharmonic functions to the date 1937, see T. Radó, "Subharmonic functions" (*Ergebnisse der Math.*).

‡ Naturally w_n is defined only for $n > 1/d\{z, F(D)\}$.

§ The difference of the values of w_n at two near z's is a multiple of the difference of the integrals of w over two small areas.

22. *Subordination.*

22.1. In what follows we shall denote the unit circle $|z| < 1$ always by γ. We generally use ρ for $|z|$, and in any case ρ will always satisfy $\rho < 1$. We denote by η any complex constant satisfying $|\eta| = 1$.

We denote by $\omega(z)$ any function, regular in γ, and satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ in γ. By Schwarz's lemma (Theorem 107) $|\omega(z)| \leqslant |z|$ in γ. *Equality holds if and only if* $\omega = \eta z$. In fact, $\Phi = \omega/z$ is regular and $|\Phi| \leqslant 1$ in γ. Hence, by the maximum modulus principle (Theorem 101), $|\Phi(z)| = 1$ for some z, z_0 say, if and only if $\Phi \equiv \Phi(z_0) = \eta$.

Let $\bar{f}(z)$ be a given function, regular, or, more generally, meromorphic in γ. If $f(z)$ is any function of the form

$$f(z) = \bar{f}\big(\omega(z)\big), \tag{1}$$

we shall say that f is "subordinate" to $\bar{f}$ in γ. It is evident that $f(0) = \bar{f}(0)$, and that f also is meromorphic in γ. We say similarly that f is subordinate to $\bar{f}$ in $|z| < R$, if $f(R\zeta)$ is subordinate to $\bar{f}(R\zeta)$ for ζ in γ.

The definition (1) may seem rather abstract; but there is a simple geometrical interpretation. Let us first suppose that $w = \bar{f}$ is a schlicht function in γ, *i.e.* that it takes no value more than once. $\bar{f}$ conformally represents γ on a simply-connected domain $\mathfrak{W} = \mathfrak{W}(\bar{f})$ (on the simple w-sphere) which does not overlap itself, or, as we shall say, "is a schlicht domain". In this case, if f is subordinate to $\bar{f}$, we have $f(0) = \bar{f}(0)$, and f takes only values inside $\mathfrak{W}$. [Note that f itself need not be schlicht.]

Conversely, if f has these properties, it is subordinate to $\bar{f}$. Let $Z(w)$ be that function, inverse to $w = \bar{f}(z)$ and so regular in $\mathfrak{W}$, for which $Z\big(\bar{f}(0)\big) = 0$. Clearly $\omega(z) = Z\big(f(z)\big)$ is regular, and $|\omega| < 1$, in γ. Also $\omega(0) = Z\big(f(0)\big) = Z\big(\bar{f}(0)\big) = 0$. Hence $f = \bar{f}(\omega)$ is subordinate to $\bar{f}$.

More generally, let $w = \bar{f}$ be "locally schlicht" in γ, *i.e.* let $\bar{f}$ have no poles of order higher than 1, and $\bar{f}' \neq 0$ in γ. $\bar{f}$ conformally represents γ on a simply connected domain $\mathfrak{W} = \mathfrak{W}(\bar{f})$ on a Riemann surface (considered on the w-sphere), the point $w_0 = \bar{f}(0)$, in some particular sheet, corresponding to $z = 0$. Let f be subordinate to $\bar{f}$. To each point z in γ corresponds a well determined point $w = \bar{f}\big(\omega(z)\big)$ of $\mathfrak{W}$. In particular $f(0) = w_0$, and to any curve in γ beginning at $z = 0$ corresponds a well determined curve in $\mathfrak{W}$ beginning at w_0: the values of $f(z)$, if continued analytically from $f(0) = w_0$, remain inside $\mathfrak{W}$. [Note again that f itself need not be locally schlicht.] Conversely, if f has these properties, it is subordinate to $\bar{f}$. The proof is the same as in the "schlicht" case.

Finally, if $\bar{f}$ is not locally schlicht, the only difference in the state of things is that the domain $\mathfrak{W}(\bar{f})$ on the Riemann surface possesses algebraic winding points in $\mathfrak{W}$. This general conception of subordination, if less easy to grasp intuitively, is clearly the inevitable extension of the simple one, for schlicht $\bar{f}$.

We begin with some simple consequences of our definition.

LEMMA 1.—*If f is subordinate to $\bar{f}$ in $|z| < R$, and $\bar{\chi}(z) = \psi\{\bar{f}(z)\}$ is meromorphic in $|z| < R$, then $\chi(z) = \psi(f)$ is (meromorphic and) subordinate to $\bar{\chi}$ in $|z| < R$.*

For $\chi = \psi(f) =$

2.—*If f is subordinate to $\bar{f}$ in $|z| < R$, and $R' < R$, then f is subordinate to $\bar{f}$ in $|z| < R'$.*

Let $\zeta = R'\xi/R$. If ξ is in γ, $|\omega(\zeta)| < R'/R$, and so $\omega(\zeta) = (R'/R)\,\omega_1(\xi)$. Hence $f(R\zeta) = \bar{f}\big(R\omega(\zeta)\big)$ becomes $f(R'\xi) = \bar{f}\big(R'\omega_1(\xi)\big)$.

The following slight generalisation of Lemma 2, based on the inequality $|\omega(z)| \leqslant |z|$ and our remark on the case of equality, is important enough to be formulated as a theorem.

THEOREM 209.—*Let $\bar{f}$ be meromorphic in γ, and let $\mathfrak{W}'_\rho$ denote the closed sub-domain of $\mathfrak{W}(\bar{f})$ corresponding to the values of $\bar{f}$ in $|z| \leqslant \rho$. If f is subordinate to $\bar{f}$ in γ, then the values of f in $|z| \leqslant \rho$ [by analytical continuation from $f(0) = \bar{f}(0) = w_0$] remain in $\mathfrak{W}'_\rho$. Frontier points of $\mathfrak{W}'_\rho$ are attained if and only if $f = \bar{f}(\eta z)$. In particular, if $\bar{f}$ (and so f) is regular in $|z| \leqslant \rho$,*

(2)
$$\operatorname*{Max}_{|z|=\rho} |f(z)| \leqslant \operatorname*{Max}_{|z|=\rho} |\bar{f}(z)|.$$

with equality if and only if $f = \bar{f}(\eta z)$.

22 . 2. We recall (Theorem 1) the results

$$M_0(\rho, f) = \lim_{\lambda \to 0} M_\lambda(\rho, f) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log|f(\rho e^{i\theta})|\,d\theta\right\},$$

$$M_\infty(\rho, f) = \lim_{\lambda \to \infty} M_\lambda(\rho, f) = M(\rho, f) = \operatorname*{Max}_{|z|=\rho} |f(z)|.$$

THEOREM 210.—*Let $\bar{g}(z) = \bar{g}(x, y)$ be subharmonic in γ, $\omega(z) = u + iv$ regular and $|\omega| \leqslant \rho = |z|$ in γ, $g(x, y) = \bar{g}(u, v)$. Then*

$$\int_{-\pi}^{\pi} g(\rho e^{i\theta})\,d\theta \leqslant \int_{-\pi}^{\pi} \bar{g}(\rho e^{i\theta})\,d\theta.$$

Let $\bar{G}(x, y) = \bar{G}(x, y; r)$ be the Poisson integral of the values of $\bar{g}$ on $|z| = r < 1$, and let $G(x, y) = \bar{G}(u, v)$. Then, by Theorem 203,

$$\bar{g}(x, y) \leqslant G(x, y) \quad (\rho < r).$$

Hence, since $|\omega| \leqslant \rho$,

$$g(x, y) = \bar{g}(u, v) \leqslant \bar{G}(u, v) = G(x, y).$$

Also G is harmonic in $\rho < r$ (or everywhere $-\infty$). Hence, if I denotes the average integral,

$$I(g, \rho) \leqslant I(G, \rho) = G(0, 0) = \bar{G}(0, 0) = I(\bar{g}, r).$$

We now make $r \to \rho + 0$ and use Theorem 207 (ii), obtaining

$$I(g, \rho) \leqslant I(\bar{g}, \rho).$$

COROLLARY.—*Let f be subordinate to $\bar{f}$ in γ. Then if $\bar{f}$ (and so f) is regular in $|z| \leqslant \rho$,*

(1) $$M_\lambda(\rho, f) \leqslant M_\lambda(\rho, \bar{f}) \quad (0 \leqslant \lambda \leqslant \infty),$$

(2) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \overset{+}{\log}|f|\, d\theta \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \overset{+}{\log}|\bar{f}|\, d\theta,$$

(3) $$M_k(\rho, \mathfrak{R}f) \leqslant M_k(\rho, \mathfrak{R}\bar{f}) \quad (1 \leqslant k \leqslant \infty).$$

For $|f|^\lambda$, $\log|f|$, $\overset{+}{\log}|f|$, $|\mathfrak{R}f|^k$ are subharmonic, and the cases $\lambda = \infty$, $k = \infty$ follow from Theorem 209.

We observe in passing that the result (Theorem 208) that M_λ is an increasing function of ρ is a particular case of the corollary. In fact, if $\rho_1 < \rho$, the obviously $f(\rho_1 z/\rho)$ is subordinate to $f(z)$, and thus

$$M_\lambda(\rho_1, f) \leqslant M_\lambda(\rho, f),$$

22 . 3. This is a convenient moment to discuss "Jensen's theorem".

THEOREM 211.—*Let $f(z)$ be meromorphic in $|z| \leqslant r$, $f(0) \neq 0, \infty$, and let $a_1, a_2, \ldots, a_n$ be the zeros, $b_1, b_2, \ldots, b_m$ the poles, of f in $|z| < r$. Then*

(1) $$\log M_0(r, f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|\, d\theta = \log\left(r^{n-m}|f(0)|\left|\frac{b_1 b_2 \ldots b_m}{a_1 a_2 \ldots a_n}\right|\right).$$

We can reduce the theorem to the special case in which, with $R \geqslant 0$, we have

(2) $$J(R) = \frac{1}{2\pi}\int \log|1 + Re^{i\theta}|\, d\theta = \overset{+}{\log} R = \begin{cases} \log R & (R \geqslant 1) \\ 0 & (R \leqslant 1). \end{cases}$$

For

$$\log|f| = \log|f_1| + \log|f(0)| + \sum_{\nu=1}^{n} \log\left|1 - \frac{z}{a_\nu}\right| - \sum_{\mu=1}^{m} \log\left|1 - \frac{z}{b_\mu}\right|,$$

where $\log f_1(z)$ is regular in $|z| \leqslant r$ and zero at $z=0$, so that

$$\frac{1}{2\pi}\int \log|f_1(re^{i\theta})|\,d\theta = \log|f_1(0)| = 0.$$

Next, the first case in (2) is reducible at once to the second by the identity

$$\log|1+Re^{i\theta}| = \log|R| + \log\left|1 + \frac{1}{R}e^{-i\theta}\right|$$

(and a trivial change of θ to $-\theta$ in the integration). Finally, the second case in (2) is trivial when $R < 1$.

Thus we have $J(R) = \overset{+}{\log} R$ unless $R = 1$, and this case $R = 1$ is the only point that cannot be disposed of trivially. There are naturally many ways of settling it, but none can be called trivial. With the theorems at our disposal we can appeal *either* to the continuity *or* the non-decreasing property of $\int_{-\pi}^{\pi} \log|f(z)|\,d\theta$, with $f(z) = 1+z$.

At an opposite extreme the result is equivalent to the definite integral

$$\int_0^{\frac{1}{2}\pi} \log \operatorname{cosec}\theta\,d\theta = \tfrac{1}{2}\pi\log 2,$$

which we may suppose known!

22.4. If $\bar{f}$, and therefore f, is regular at $z=0$, they are regular in some circle γ'_ρ, or $|z| \leqslant \rho$. We then have expansions

(1) $$f(z) = \sum_0^\infty a_n z^n, \quad \bar{f}(z) = \sum_0^\infty \bar{a}_n z^n,$$

valid in γ'_ρ. We have, of course, $a_0 = \bar{a}_0$.

THEOREM 212.—*Let f be subordinate to $\bar{f}$ in γ, and $\bar{f}$ regular at $z=0$. Then*

(2) $$|a_1| \leqslant |\bar{a}_1|,$$

with equality if and only if $f = \bar{f}(\eta z)$. Also

(3) $$|a_2| \leqslant \operatorname{Max}(|\bar{a}_1|, |\bar{a}_2|).$$

We have $f = \bar{f}(\omega)$. Let $\omega = \alpha_1 z + \alpha_2 z^2 + \ldots$. Obviously $|\alpha_1| \leqslant 1$, with equality if and only if $\omega = \eta z$. Hence

$$|a_1| = |f'(0)| = |\bar{f}'(0)||\omega'(0)| = |\bar{a}_1||\alpha_1| \leqslant |\bar{a}_1|,$$

with equality if and only if $f = \bar{f}(\eta z)$.

Next, let $|\alpha_1| < 1$. The function

$$\omega_1 = \frac{\omega/z - \alpha_1}{(\omega/z)\,\alpha_1' - 1} = \frac{\alpha_2}{|\alpha_1|^2 - 1} z + \ldots$$

is also of the type ω. Hence $|\alpha_2| \leqslant 1 - |\alpha_1|^2$, and this remains true if $|\alpha_1| = 1$ (when $\omega = \eta z$, $\alpha_2 = 0$). Now $f'' = \bar{f}''(\omega)\,\omega'^2 + \bar{f}'(\omega)\,\omega''$, and so

$$|a_2| \leqslant |\bar{a}_2||\alpha_1|^2 + |\bar{a}_1|\,(1 - |\alpha_1|^2) \leqslant \operatorname{Max}\{|\bar{a}_1|,\ |\bar{a}_2|\}.$$

We note explicitly the special cases $\lambda = 0,\ 2,\ \infty$ of Theorem 210, Corollary. They give respectively, for $\bar{f}$ regular in γ_ρ',

$$(4) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(\rho e^{i\theta})|\,d\theta \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|\bar{f}(\rho e^{i\theta})|\,d\theta,$$

$$(5) \qquad \Sigma\,|a_n|^2 \rho^{2n} \leqslant \Sigma\,|\bar{a}_n|^2 \rho^{2n},$$

$$(6) \qquad \operatorname*{Max}_{|z|=\rho} |f| \leqslant \operatorname*{Max}_{|z|=\rho} |\bar{f}|,$$

the last of which we have had already.

The case $\lambda = 1$ is also important, since it provides inequalities for the coefficients of f.

THEOREM 213.—*If $\bar{f}$ is regular and f is subordinate to $\bar{f}$ in γ, then, for $n > 1$,*

$$|a_n| \leqslant e\left[\frac{M_1(\rho, \bar{f})}{\rho}\right]_{\rho=(n-1)/n}.$$

For

$$|a_n| = \frac{\rho^{-n}}{2\pi}\left|\int_{-\pi}^{\pi} f(\rho e^{i\theta})\,e^{-ni\theta}\,d\theta\right| \leqslant \frac{\rho^{-n}}{2\pi}\int_{-\pi}^{\pi} |f(\rho e^{i\theta})|\,d\theta \leqslant \rho^{-(n-1)}\,\frac{M_1(\rho, \bar{f})}{\rho}.$$

In this we take $\rho = \dfrac{n-1}{n}$, and observe that $\rho^{-(n-1)} = \left(1 + \dfrac{1}{n-1}\right)^{n-1} < e$.

THEOREM 214.—*Let $\bar{f}$ be regular and f subordinate to $\bar{f}$ in γ, and $f(0) = \bar{f}(0) = 0$. Let $z_n(a)$, $\bar{z}_n(a)$ be the n-th non-zero roots of $f - a = 0$ and $\bar{f} - a = 0$ arranged in order of increasing moduli;*

$$|z_n(a)| = \rho_n(a), \quad |\bar{z}_n(a)| = \bar{\rho}_n(a).$$

Then

$$\prod_{\rho_n(a)<\rho}\left(\frac{\rho}{\rho_n(a)}\right)\leqslant\prod_{\bar\rho_n(a)<\rho}\left(\frac{\rho}{\bar\rho_n(a)}\right)\quad(a\neq 0),$$

$$|a_\nu|\,\rho^{\nu-1}\prod_{\rho_n(0)<\rho}\left(\frac{\rho}{\rho_n(0)}\right)\leqslant|\bar a_\mu|\,\rho^{\mu-1}\prod_{\bar\rho_n(0)<\rho}\left(\frac{\rho}{\bar\rho_n(0)}\right),$$

where a_ν is the first non-zero coefficient of f, and $\bar a_\mu$ that of $\bar f$. If product contains no factors it is interpreted to mean unity

Since $f-a$ is subordinate to $\bar f-a$, we have, on every circle $|z|=\rho$,

(7) $$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f-a|\,d\theta\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|\bar f-a|\,d\theta.$$

By Theorem 211, if $\phi=\sum_\lambda^\infty b_n z^n$ is regular in γ, $b_\lambda\neq 0$, and ζ_n is the n-th non-zero root of $\phi=0$, we have

(8) $$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|\phi(\rho e^{i\theta})|\,d\theta=\log\left|\,b_\lambda\rho^\lambda\prod_{|\zeta_n|<\rho}\left(\frac{\rho}{\zeta_n}\right)\right|.$$

From (7) and (8) we obtain Theorem 214.

Further inequalities for the coefficients a_n.

22.5. The inequality (5) of § 22.4 can be generalised. We have in fact

THEOREM 215.—*If f is subordinate to $\bar f$ in γ and $\bar f$ is regular at $z=0$, then*

(1) $$\sum_1^n|a_k|^2\leqslant\sum_1^n|\bar a_k|^2\quad(n=1,2,\ldots).$$

Let $f=\bar f(\omega)$, and let

$$s_n(z)=\sum_1^n a_k z^k,\quad \bar s_n(z)=\sum_1^n\bar a_k z^k.$$

Then $\bar s_n(z)$, $\bar s_n(\omega)$ are regular in $|z|<1$, and we have, for $|z|<1$,

$$\bar s_n(\omega)=\sum_1^n\bar a_k\omega^k=s_n(z)+\sum_{n+1}^\infty b_k z^k,$$

say. Applying (5) of § 22.4, with $\bar s_n$ for $\bar f$ and any $\rho<1$, we obtain

$$\sum_1^n|a_k|^2\rho^{2k}\leqslant\sum_1^n|a_k|^2\rho^{2k}+\sum_{n+1}^\infty|b_k|^2\rho^{2k}\leqslant\sum_1^n|\bar a_k|^2\rho^{2k},$$

and in this we make $\rho\to 1$.

From (1) we have at once

$$(2) \qquad |a_n| \leqslant \sqrt{n}\,\mathrm{Max}\,(|\bar{a}_1|, |\bar{a}_2|, \ldots, |\bar{a}_n|).$$

In particular, if $\bar{a}_n = O(1)$, then $a_n = O(\sqrt{n})$.

More than this is not true in the general case; for example (although we cannot prove it here) there exist a regular $\bar{f}$ and a subordinate f, such that $\bar{a}_n = O(1)$ and a_n is not $o(\sqrt{n})$. But *if the behaviour of* $\bar{a}_n$ *is sufficiently regular* (and $\bar{a}_n$ is increasing) we can actually attain the ideal "$|a_n| \leqslant |\bar{a}_n|$".

THEOREM 216.—*Let f be subordinate to $\bar{f}$ in γ, where $\bar{f}$ is regular at $z = 0$, and let $n > 2$.*

(i) *If the set of numbers $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ is non-negative, non-increasing, and convex; i.e.*

$$\bar{a}_n \geqslant 0; \quad \bar{a}_{n-1} - \bar{a}_n \geqslant 0; \quad \bar{a}_k - 2\bar{a}_{k+1} + \bar{a}_{k+2} \geqslant 0 \quad (1 \leqslant k \leqslant n-2);$$

then $|a_n| \leqslant \bar{a}_1$

(ii) *If the set of numbers $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ is non-negative, non-decreasing, and convex; i.e.*

$$\bar{a}_1 \geqslant 0; \quad \bar{a}_2 - \bar{a}_1 \geqslant 0; \quad \bar{a}_k - 2\bar{a}_{k-1} + \bar{a}_{k-2} \geqslant 0 \quad (3 \leqslant k \leqslant n);$$

then $|a_n| \leqslant \bar{a}_n$.

The proof depends on two lemmas, the first of which is of interest in itself.

LEMMA 3.—*Let $g(z) = \tfrac{1}{2}b_0 + \sum_1 b_k z^k$, where, for $k \geqslant 0$,*

$$b_k \geqslant 0; \quad \Delta_k = b_k - b_{k+1} \geqslant 0; \quad \Delta_k^2 = b_k - 2b_{k+1} + b_{k+2} \geqslant 0.$$

Then $\mathfrak{R} g(z) \geqslant 0$ in γ.

Let $t_0 = \tau_0 = \tfrac{1}{2}$, and

$$t_n(z) = \tfrac{1}{2} + \sum_1^n z^k; \quad \tau_n(z) = \sum_0^n t_k(z) \quad (n \geqslant 1).$$

With the notation of § 5.31 (3), $\mathfrak{R}\tau_n(z) = \tfrac{1}{2}R_n(t) \geqslant 0$ in γ. Also as $n \to \infty$ $t_n(z) \to \tfrac{1}{2}(1+z)/(1-z)$, the real part of which is non-negative in γ. Summing twice by parts, we have

$$\tfrac{1}{2}b_0 + \sum_1^n b_k z^k = \sum_0^n \Delta_k t_k(z) + b_{n+1} t_n(z) = \sum_0^n \Delta_k^2 \tau_k(z) + \Delta_{n+1}\tau_n(z) + b_{n+1} t_n(z).$$

Hence $\mathfrak{R}\left\{\frac{1}{2}b_0+\sum_1^n b_k z^k\right\} \geqslant b_{n+1}\mathfrak{R}t_n(z)$ in γ, and the lemma follows when we make

LEMMA 4.—*Let $k \geqslant 1$ and $\omega^k = \sum_{n=k}^{\infty} a_n^{(k)} z^n$. Then in γ*

$$|P_n(z)| = \left|\sum_{k=1}^{n} a_n^{(k)} z^k\right| \leqslant 1.$$

We begin with the following general remark. If $f=\bar{f}(\omega)$, then

$$a_n = \sum_{k=1}^{n} a_n^{(k)} \bar{a}_k. \tag{3}$$

Now let $|\zeta| \leqslant 1$. The function $h(z)=\frac{1}{2}(1+\zeta\omega)/(1-\zeta\omega)$ has a non-negative real part in γ. By (3), the n-th coefficient of h is $P_n(\zeta)$, and the lemma follows from Theorem 110.

Proof of Theorem 216.—(i) By Lemma 3,

$$g(z) = \tfrac{1}{2}\bar{a}_1+\bar{a}_2 z+\ldots+\bar{a}_{n-1}z^{n-2}+\bar{a}_n \sum_{n-1}^{\infty} z^k$$

has a non-negative real part in γ. Also, by § 10.4 (2),

$$\bar{a}_k \rho^{k-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathfrak{R}g(\rho e^{i\theta})\, e^{-i(k-1)\theta}\, d\theta \quad (1 \leqslant k \leqslant n).$$

Hence, using Lemma 4,

$$\left|\sum_{k=1}^{n} a_n^{(k)}\bar{a}_k \rho^{k-1}\right| \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} \mathfrak{R}g(\rho e^{i\theta})\,|e^{i\theta} P_n(e^{-i\theta})|\, d\theta \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} \mathfrak{R}g(\rho e^{i\theta})\, d\theta = \bar{a}_1.$$

Let $\rho \to 1$, when (i) follows from (3).

(ii) By Lemma 3,

$$h(z) = \tfrac{1}{2}\bar{a}_n+\bar{a}_{n-1}z+\ldots+\bar{a}_2 z^{n-2}+\bar{a}_1 \sum_{n-1}^{\infty} z^k$$

has a non-negative real part in γ. Also

$$\bar{a}_k \rho^{n-k} = \frac{1}{\pi}\int_{-\pi}^{\pi} \mathfrak{R}h(\rho e^{i\theta})\, e^{-i(n-k)\theta}\, d\theta \quad (1 \leqslant k \leqslant n).$$

Hence

$$\left|\sum_{k=1}^{n} a_n^{(k)}\bar{a}_k \rho^{n-k}\right| \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} \mathfrak{R}h(\rho e^{i\theta})\,|e^{-in\theta} P_n(e^{i\theta})|\, d\theta \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} \mathfrak{R}h(\rho e^{i\theta})\, d\theta = \bar{a}$$

and the proof is completed as before by making $\rho \to 1$.

Lemma 4 enables us to give a proof of Theorem 213 independent of Theorem 210. By Lemma 2, $f(\rho z)$ is subordinate to $\bar{f}(\rho z)$, so that, for some ω, $f(\rho z)=\bar{f}(\rho\omega)$. Also the n-th coefficients of $f(\rho z)$ and $\bar{f}(\rho z)$ are $a_n\rho^n$ and $\bar{a}_n\rho^n$, respectively. Now

$$\bar{a}_k\rho^k=\frac{1}{2\pi}\int_{-\pi}^{\pi}\bar{f}(\rho e^{i\theta})\,e^{-ik\theta}\,d\theta,$$

and so, by (3) and Lemma 4,

$$|a_n\rho^n|=\left|\sum_1^n \alpha_n^{(k)}\bar{a}_k\rho^k\right|=\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\bar{f}(\rho e^{i\theta})\,P_n(e^{-i\theta})\,d\theta\right|\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}|\bar{f}(\rho e^{i\theta})|\,d\theta.$$

23. *The principle of subordination.*

23.1. The general theory of subordination is a powerful weapon for dealing with the problem of the set of values taken (or omitted) by an analytic function. We may look at the question from two different points of view.

I. In the first place let $\bar{h}(z)$ be a given function, meromorphic in γ, and consider the class $\mathfrak{H}$ of all functions $h(z)$, meromorphic and subordinate to $\bar{h}(z)$ in γ. In order to apply the general theory of subordination to the class $\mathfrak{H}$ we begin with a special study of the superordinate function $\bar{h}$. If we suppose, to fix ideas, that our knowledge of it is complete, we obtain results of several distinct types.

A. We know the domain $\mathfrak{W}_\rho(\bar{h})$ in which the values of each h for $|z|<\rho$ must lie [Theorem 209]. In particular, if $\bar{h}$ (and so every h) is regular in $|z|\leqslant\rho$, we obtain an upper bound for $|h|$ in $|z|\leqslant\rho$. More generally, under the same assumption, we obtain upper bounds for the means $M_\lambda(\rho, h)$ [Theorem 210, Corollary]. All these results are "best possible" ones.

B. Assuming again $\bar{h}$ to be regular at $z=0$, and writing $h(z)=\sum\limits_0^\infty h_n z^n$, $h_0=\bar{h}_0$, we obtain upper bounds for $|h_n|$, for $\sum\limits_0^n|h_k|^2$, and for $\sum\limits_0^\infty|h_n|^2\rho^{2n}$ [§22.4, 5].

C. Lower bounds for the products $\prod\limits_{\rho_n<\rho}\rho_n(a)$, where $\rho_n=|z_n|\neq 0$ $h(z_n)=a$, are given by Theorem 214.

II. The second point of view is to start with a given function $f(z)$ [or with a class $\mathfrak{F}$ of such functions], meromorphic in γ. Any subordination of f to some other function F would imply certain consequences, of the types A, B, C above, for instance. If we now know, by some given property of f [or of the class $\mathfrak{F}$], that one of these consequences is impossible, then the subordination to F is impossible, and $\mathfrak{W}(f)$ cannot "lie upon" $\mathfrak{W}(F)$. This can be important *positive* information; for example, *if F is schlicht, f must take values that F does not.* And (as will be amply illustrated below) F is largely at our disposal.

The systematic application of these ideas in both directions (I and II) is what we call the "principle of subordination" [Lindelöf principle]. It may be regarded as an extension of the simple "maximum modulus principle" of § 9. The most interesting and important applications are those connected with the famous Theorem of Picard [§ 24.4], in which the elliptic modular functions are the superordinate functions. We postpone these applications, however, to later sections, confining ourselves here to more elementary and more general results.

23.2. We denote throughout by $\mathcal{M}$ the (very general) class of functions $f(z)$, meromorphic in γ and satisfying

(1) $$f(0) = 0, \quad f'(0) = 1,$$

and by $\mathcal{R}$ the sub-class of $\mathcal{M}$ consisting of those functions of $\mathcal{M}$ that are regular in γ.

Our first theorem is an immediate consequence of Theorem 212.

THEOREM 217.—*Let $0 < t \leqslant 1$, $0 < \rho \leqslant 1$. If $f(z)$ and $F(z)$ are functions of $\mathcal{M}$, then $f(\rho z)$ is not subordinate to $tF(\rho z)$, except in the case $t = 1$, $f(z) \equiv F(z)$.*

This theorem, simple as it looks, is of great importance in direction II of our principle. Given an f of $\mathcal{M}$, we can, by choosing suitable functions $F(z)$, or rather suitable Riemann domains $\mathfrak{W}(F)$, obtain information about the values taken by f. It may be noted that if $\mathfrak{W}(F)$ is the domain of F of $\mathcal{M}$, the "rotated" domain $\eta\mathfrak{W}(F)$ (with $|\eta| = 1$) belongs to the function $\eta F(\eta^{-1}z)$ of $\mathcal{M}$.

23.3. The simplest applications of our principle come by choosing the function $F(z)$ to be z, and we devote the present sub-section (§ 23.3) to it. The class of functions subordinate to z in γ is the class of functions $\omega(z)$ (of § 22). Here the elementary inequalities $|a_n| \leqslant 1$, and, more generally,

(1) $$\sum_1^\infty |a_n|^2 \leqslant 1,$$

are the simplest examples of type I, B. For a result of type I, C, we have

THEOREM 218 (Jensen).—*Let* $f(z) \not\equiv 0$ *be regular and* $|f(z)| \leqslant M$ *in* γ. *Let* (ρ_n) *be the sequence of the moduli of the zeros* $z_n \neq 0$ *of* $f(z)$ *in* γ, *counted according to multiplicity and arranged in increasing order. Then*

$$\Pi \rho_n \geqslant |a_\nu| M^{-1}, \tag{2}$$

where a_ν *is the first non-zero coefficient of the power series of* f *in* γ.

In particular, $\rho_n \geqslant |a_\nu| M^{-1}$ *for each such* ρ_n.

COROLLARY.—*If* $f(z)$ *is regular in* γ, *or, more generally, is meromorphic in* γ *and subordinate to a function* $\bar{f}(z)$ *for which* $\Pi \bar{\rho}_n(0) > 0$; *and if there exists a sequence of zeros* $\neq 0$ *of* $f(z)$ *such that* $\Pi \rho_n = 0$; *then* $f(z) \equiv 0$.

If no ρ_n exists, (2) is simply Cauchy's inequality for $|a_\nu|$. If there are at least k such ρ_n, and if $p \geqslant k$, we observe that $zf(z)$ is subordinate to Mz in γ, so that, by the second inequality of Theorem 214, if ρ is near enough to 1,

$$|a_\nu| \rho^\nu \frac{\rho^p}{\prod\limits_1^p \rho_n} \leqslant |a_\nu| \rho^\nu \prod_{\rho_n < \rho} \left(\frac{\rho}{\rho_n}\right) \leqslant M.$$

We obtain (2) by making first $\rho \to 1$ and then $p \to \infty$.

The corollary is an immediate consequence of Theorem 214 if f is regular. In the general case we have $f = \bar{f}(\omega)$. Denote by $\{\omega^{(i)}\}$ the sequence of zeros of $\bar{f}$ in γ, each counted *once* only, and arranged by increasing moduli. Since $f(z_n) = \bar{f}\{\omega(z_n)\} = 0$, the values $\omega(z_n)$ are among the $\omega^{(i)}$. Let $z_n^{(i)}$ denote those z_n, if any, for which $\omega(z_n^{(i)}) = \omega^{(i)}$. Suppose that $\omega \not\equiv 0$. If $z_n^{(i)}$ is a zero of order $k_n^{(i)}$ for f and of order $\kappa_n^{(i)}$ for $\omega(z) - \omega_i$, and $\omega^{(i)}$ is a zero of $\bar{f}$, order $l^{(i)}$, then clearly

$$k_n^{(i)} = l^{(i)} \kappa_n^{(i)}. \tag{3}$$

Now ω is subordinate in γ to z. If one of the $\omega^{(i)}$, $\omega^{(1)}$ say, is zero (that is, if $\bar{f}(0) = 0$), and if a_ν is the first non-zero coefficient of ω, then by the second inequality of Theorem 214

$$\Pi' \rho_n^{(1)} \geqslant |a_\nu|,$$

where each $\rho_n^{(1)}$ occurs $\kappa_n^{(1)}$ times in the product. Hence, by (3),

$$\Pi \rho_n^{(1)} \geqslant |a_\nu|^{l^{(1)}}, \tag{4}$$

where now $\rho_n^{(1)}$ occurs according to multiplicity $k_n^{(1)}$.

Similarly, if $\omega^{(i)} \neq 0$, the first inequality of Theorem 214 yields

$$\Pi' \rho_n^{(i)} \geqslant |\omega^{(i)}|,$$

where each $\rho_n^{(i)}$ occurs $\kappa_n^{(i)}$ times in the product. Hence

(5) $$\Pi \rho_n^{(i)} \geqslant |\omega^{(i)}|^{l^{(i)}},$$

where now $\rho_n^{(i)}$ occurs according to multiplicity $k_n^{(i)}$. Combining (4) and (5) we obtain

(6) $$\Pi \rho_n(0) = \prod_i \Pi \rho_n^{(i)} \geqslant |\alpha_\nu|^{l^{(1)}} \prod_i |\omega^{(i)}|^{l^{(i)}} \geqslant |\alpha_\nu|^{l^{(1)}} \Pi \bar{\rho}_n(0) > 0.$$

This holds also, with $|\alpha_\nu|^{l^{(1)}} = 1$, if no $\omega^{(i)}$ vanishes. Since (6) contradicts the hypothesis of the Corollary, $\omega(z) \equiv 0$; that is, $f(z) \equiv \bar{f}(0)$ is a constant, which must be zero.

The corollary leads to an important extension of Vitali's Theorem 113. If we examine the proof in § 10.6 it will appear that the essential fact about the sequence $\{z_n\}$ is that a regular ϕ vanishing at every z_n must vanish identically (a "uniqueness" theorem). Corresponding to any such type of sequence we may hope to find an extension of Vitali's theorem, and for the type in the corollary we have at once the following one.

THEOREM 113*.—*Suppose* (i) f_n *regular and uniformly bounded in* γ; (ii) $f_n \to$ *a limit (necessarily finite) for each* z *of* $z_1, z_2, \ldots$, *an infinite sequence (of different* z's $\neq 0$) *in* γ *such that* $\Pi \rho_n = 0$. *Then* $f_n \to f$ *in* γ, *uniformly in every circle* $|z| \leqslant \rho < 1$.

To complete the results derived from the case $F = z$, we observe that any (other) f of $\mathcal{M}$ must take, in $|z| \leqslant \rho$, values not taken there by F: we thus have (a result of type II)

every function f *of* $\mathcal{M}$ *satisfies*

(7) $$\operatorname*{Max}_{|z|=\rho} |f| \geqslant \rho \quad (0 < \rho < 1),$$

equality being attained only if $f \equiv z$.

23.4. Consider next the class B of all functions $w = B(z) = \sum_0^\infty b_n z^n$, regular in γ and *taking there no value* w *for which* $\Re w = 0$. Clearly either $\Re w > 0$ or $\Re w < 0$ (according as $\Re b_0 \gtrless 0$) in γ, and $B(z)$ is subordinate to

(1) $$\bar{w} = \bar{B}(z) = \beta_0 \frac{1+z}{1-z} + i\gamma_0 \quad (b_0 = \beta_0 + i\gamma_0).$$

The domain $\mathfrak{W}(\bar{B})$ is the half-plane $\Re \bar{w} \gtrless 0$ according as $\beta_0 \gtrless 0$. The domain $\mathfrak{W}_\rho(\bar{B})$ [of Theorem 209] is the open circle extended over the stretch

(2) $$\beta_0 \frac{1-\rho}{1+\rho} + i\gamma_0, \quad \beta_0 \frac{1+\rho}{1-\rho} + i\gamma_0$$

as diameter; the values of each function $B(z)$ for $|z| < \rho$ must lie in this circle. In particular

(3) $$|B(z)| \leqslant |b_0| + \frac{2|\beta_0|\rho}{1-\rho} \leqslant |b_0| \frac{1+\rho}{1-\rho}.$$

This is the best possible form of Theorem 109 (with $U = 0$).

As a (best possible) result of type I, B we have, by Theorem 216 (*either* part!)

(4) $$|b_n| \leqslant 2\beta_0 \quad (n > 0) \quad \text{[Theorem 110]}.$$

23.5. Let $a \geqslant 0$, and let us consider the function

(1) $$\bar{w} = F_a(z) = \frac{z}{az+1} = z - az^2 + a^2 z^3 - + \ldots.$$

$F_a(z)$ belongs to $\mathfrak{M}$ and represents γ conformally on the interior (when $0 \leqslant a < 1$) or exterior (when $a > 1$) of the circle K_a on the stretch $-(1-a)^{-1}$, $(1+a)^{-1}$ as diameter. If $a = 1$, K_1 degenerates into the half-plane $\Re \bar{w} < \frac{1}{2}$.

If $a \neq 1$, the centre c of K_a and its radius r are given by

(2) $$c = \frac{a}{a^2-1}, \quad r = \frac{1}{|a^2-1|},$$

whence

(3) $$c^2 = \begin{cases} r(r-1) & (a < 1) \\ r(r+1) & (a > 1). \end{cases}$$

Taking all functions $\eta F_a(\eta^{-1} z)$ into account, Theorem 217 now readily yields

THEOREM 219.—(i) *Let c be an arbitrary complex number. Every function $f(z)$ of $\mathfrak{M}$ takes some value w with*

(4) $$|w-c| > \tfrac{1}{2} + \sqrt{(\tfrac{1}{4} + |c|^2)} = r_1(c),$$

and, if $c \neq 0$, also some w with

(5) $$|w-c| < -\tfrac{1}{2} + \sqrt{(\tfrac{1}{4} + |c|^2)} = r_2(c).$$

In each case the function $F(z)$ must be excepted that maps γ on the circle complementary to (4) *or* (5).

(ii) *Let ϑ be real. Every function $f(z)$ of $\mathfrak{M}$ takes some value w satisfying*

(6) $$\mathfrak{R}(we^{i\vartheta}) > \tfrac{1}{2}.$$

The function $F(z) = z/(ze^{i\vartheta}+1)$ is excepted.

Theorem 219 is a special case $(n=1)$ of

THEOREM 220.—*Let $f(z)$ be meromorphic in γ, and for small $|z|$*

(7) $$f(z) = \sum_1^\infty a_n z^n.$$

Let $C_r(c)$ denote the open circle with centre c and radius r.

(i) *Let $|c| < r$. If the values of f (regular in γ) lie entirely in $C_r(c)$, then*

(8) $$|a_n| \leqslant \frac{r^2-|c|^2}{r}.$$

(ii) *Let $|c| > r$. If f takes no values belonging to $C_r(c)$, then*

(9) $$|a_n| \leqslant \frac{|c|^2-r^2}{r}\left(\frac{|c|}{r}\right)^{n-1}$$

If, in this case, $f(z)$ has a pole ζ in γ, then

(10) $$|\zeta| \geqslant \frac{r}{|c|}.$$

We may assume $c \geqslant 0$. Put $a = c/r$ and

$$c^* = \frac{a}{a^2-1}, \quad r^* = \frac{1}{|a^2-1|}.$$

Then $c = \lambda c^*$, $r = |\lambda| r^*$, and $\lambda \gtrless 0$ according as $c \gtrless r$. The relation (3) between c^* and r^* gives

$$c^2 = r(r+\lambda); \quad \lambda = \frac{c^2-r^2}{r}.$$

In both cases, (i) and (ii), $f(z)$ is subordinate to

$$\lambda F_a(-z) = -\lambda[z+az^2+a^2z^3+\ldots].$$

Now $a < 1$ if $c < r$, and the sequence $1, a, a^2, \ldots$ is non-negative, non-increasing, and convex. Hence $|a_n| \leqslant |\lambda|$, by Theorem 216 (i). This proves (8).

Similarly $a > 1$ if $c > r$; in this case the above sequence is positive, increasing, and convex. Hence (9) follows from Theorem 216 (ii).

Finally, in this case, if ζ is a pole of f, (9) yields

$$|\zeta| \geqslant [\overline{\lim}\, |a_n|^{1/n}]^{-1} \geqslant \frac{r}{|c|}.$$

There exists a simple direct proof for (10) which is worth noting, since the idea of it can clearly be applied in similar cases. We may suppose again that $c \geqslant 0$. The function $F_a(z)$ has its pole at $\zeta^* = -1/a = -r/c$. Since f is subordinate to λF_a, the value $f(\zeta) = \infty$ must belong to $\lambda \mathfrak{W}_{|\zeta|}(F_a)$. Hence

$$|\zeta| \geqslant |\zeta^*| = \frac{r}{c}.$$

Theorem 219 may be regarded as a generalization of the classical

THEOREM 221 (Casorati-Weierstrass).—*Any non-constant function, meromorphic in the whole plane, takes values in every circle $C_r(c)$.*

To deduce this from Theorem 219 we may assume, without loss of generality, that $f(0) = 0, f'(0) = 1$, and that $c \neq 0$. Then $R^{-1}f(Rz)$ belongs to $\mathfrak{M}$ for every $R > 0$. Hence, by (5) (with c/R in place of c), we find that f takes some value in the circle with centre c and radius $2Rr_2(|c|/R)$, say. Since this radius tends to zero as $R \to \infty$, we obtain the desired result.

23.6. In this sub-section we give some more results of interest obtained by various choices of the domain $\mathfrak{W}(F)$. We omit the elementary and straightforward calculations necessary to show that the corresponding F belongs to $\mathfrak{M}$. The F are all schlicht, and then, by Theorem 217, *every function f of $\mathfrak{M}$ (except F) takes some value outside $\mathfrak{W}(F)$.* In particular, if $\mathfrak{W}(F)$ is the whole plane slit along certain curves, this means that any f takes some value on the slits.

(i) *A strip bounded by two parallel lines, containing $w = 0$, of width $\Delta \geqslant \frac{1}{2}\pi$ and eccentricity (distance of the mid-line from 0)*

$$E = \frac{\Delta}{\pi} \text{ arc cos } \frac{\pi}{2\Delta}.$$

(ii) *The whole plane slit along the part $|w| \geqslant \frac{1}{2}$ of some straight line meeting the circle $|w| = \frac{1}{2}$.*

(iii) *The whole plane slit along the parts $|w| \geqslant \sqrt[n]{(\frac{1}{4})}$ of n symmetric rays from* 0.

(iv) *The whole plane slit along the stretch c, $c/(1-4|c|)$, where $0 < |c| < \frac{1}{4}$.*

(v) *The whole plane slit along an arc of length* 4 arc sin ρ *on the circle $|w| = \rho$, $0 < \rho < 1$.*

In example (iii), with $n=1$, the function

$$(1) \qquad K(z)=\frac{z}{(1-z)^2}=z+2z^2+3z^3+\ldots$$

of $\mathfrak{M}$ represents γ on the whole plane slit along the part $w \leqslant -\frac{1}{4}$ of the negative real axis. In fact

$$\left(K(z)+\tfrac{1}{4}\right)^{\frac{1}{2}}=\tfrac{1}{2}\,\frac{1+z}{1-z},$$

where the right-hand side has its real part positive in γ. The function $K(z)$ plays a fundamental role in the theory of "schlicht" functions (see § 26 below).

If a function f is subordinate to K, i.e. if f takes no values $w \leqslant -\frac{1}{4}$, then $|a_n| \leqslant n$. This follows, for instance, from Theorem 216 (ii).

23.7. The following example is so important for the general applications of the principle of subordination that we state it as a theorem.

THEOREM 222 (Landau).—*Let $M \geqslant 1$. If $f(z)$ of $\mathfrak{R}$ satisfies $|f|<M$ in γ, then f takes all values w_0 with $|w_0| \leqslant \tau(M)$ where*

$$(1) \qquad \tau(M)=Me^{-x}; \quad \frac{\sinh x}{x}=M \quad (x>0).$$

If $|w_0|=\tau(M)$, that function F of $\mathfrak{R}$ must be excepted which represents γ conformally on the part $|w|<M$ of the Riemann surface of $\log(w-w_0)$†.

We may suppose w_0 real and positive. We begin by determining the function F, which we do by easy stages; the calculations, if a little tiresome, are forced. The function

$$F_1=\frac{z}{1-z}$$

belongs to $\mathfrak{R}$, $\mathfrak{W}(F_1)$ being the half-plane $\mathbf{R}w>-\frac{1}{2}$. Hence, if a is a positive constant to be determined later,

$$F_2=a-ae^{-F_1/a}$$

belongs to $\mathfrak{R}$, never takes the value a, and has for $\mathfrak{W}(F_2)$ the part $|w-a|<ae^{1/(2a)}$ of the Riemann surface of $\log(w-a)$. We can now

† The circle $|w|<M$ with w_0 excluded is a doubly connected surface; the surface of the theorem, with its infinite winding-point at w_0, is simply connected. (An infinite winding-point is a boundary point of a surface, and does not belong to it.)

modify F_2 by the linear transformation

(2) $$F = \frac{\lambda F_2}{F_2 + \lambda}.$$

F belongs to $\mathfrak{R}$ if F_2 never takes the value $-\lambda$, and contains the undetermined a and λ. We choose λ so as to make $\zeta = \frac{\lambda\zeta'}{\zeta'+\lambda}$ describe a circle $|\zeta| =$ constant when ζ' describes $|\zeta' - a| = ae^{1/(2a)}$; this happens, with $|\zeta| = M = \sinh\{1/(2a)\}$, if $\lambda = a(e^{1/a}-1)$, and then F [given by (2)] never takes the value $\frac{\lambda a}{a+\lambda}$ (corresponding to the impossibility $F_2 = a$), nor F_2 the value $-\lambda$ (corresponding to $F_1 \neq -1$). We choose a so that $\tau = \frac{\lambda a}{a+\lambda} = a(1-e^{-1/a}) = Me^{-1/(2a)}$, and then F, now completely determined, has the required properties; viz., it belongs to $\mathfrak{R}$, and represents γ conformally on the part $|w| < M$ of the Riemann surface of $\log(w-\tau)$, M being connected with τ by the relations (1), with $x = 1/(2a)$.

Now this Riemann surface $\mathfrak{W} = \mathfrak{W}(F)$ is locally schlicht, *i.e.* has no *finite* winding points, but only an infinite (logarithmic) one, and its boundaries are the same in every sheet. *It is evident, as a result of these two properties, that any function ϕ with $\phi(0) = 0$, and taking no value not in $\mathfrak{W}$ (i.e. not taking the value τ and not taking any value w with $|w| \geqslant M$), must have its values, continued analytically from $\phi(0) = 0$, confined to the surface $\mathfrak{W}$, and so, by* § 22 . 1, *be subordinate to F.* (It is clear that this inference, about surfaces $\mathfrak{W}$ that are both locally schlicht and "alike in every sheet", can be widely generalized, though we shall not attempt to state the conditions with complete precision. We shall meet an even more important instance than the present one in the "Picard" section, § 24.) Now we know that f (of $\mathfrak{R}$, satisfying $|f| < M$, and other than F) is not subordinate to F, and we conclude (since $|f| < M$) that f must take the value τ in γ.

The rest of the theorem is now easy. It is evident that M increases from 1 to ∞ as x increases from 0, while $\tau = \tau(M)$ decreases from 1 to 0. Since M can be increased arbitrarily in the hypothesis of the theorem, we see that f takes all values $\tau(M')$ for $M' \geqslant M$, and so all values $0 \leqslant w_0 \leqslant \tau$.

23 . 8. The importance of Theorem 222 for our general principle is shown by

THEOREM 223.—*Let f and F be two functions of $\mathfrak{M}$. Let $M \geqslant 1$ and f be subordinate to MF. Then f takes all values belonging to $M\mathfrak{W}_{\tau(M)/M}(F)$, where $\tau(M)$ is defined in* § 23 . 7 (1).

We have $f = MF(\omega)$, from which it follows that $M\omega(z)$ belongs to $\mathfrak{R}$. Also $M|\omega(z)| < M$ in γ. Hence $\omega(z)$ takes all values $|w| < \tau(M)/M$, and this proves the theorem.

Note that $\tau(M)/M$ decreases from 1 to 0 as $M \geqslant 1$ increases.

If now f has appropriate properties, Theorem 223 gives us lower bounds for the $M \geqslant 1$ for which f can be subordinate to MF. In particular, if f belongs to $\mathfrak{R}$ while F has poles, then $\mathfrak{W}_{\tau(M)/M}(F)$ must not contain the point $w = \infty$. Thus we obtain

THEOREM 224.—*Let f belong to $\mathfrak{R}$, and let F of $\mathfrak{M}$ have a pole ζ in γ (which we suppose that of smallest modulus). Let*

(1) $$|\zeta| = e^{-X}, \quad M^* = \frac{\sinh X}{X}.$$

If now $M < M^*$, then f is not subordinate to MF.

In particular, if F is schlicht, f takes some value not taken by MF.

This represents the improvement in the conclusion of Theorem 217 arising from the extra hypotheses that F has ζ as a pole, and f is regular instead of meromorphic. The value of the improvement is seen from the fact that *when F has only the single pole ζ in γ the result of Theorem* 224 *is best possible.*

To see that this is so, let G be the extremal function of Theorem 222 that belongs to $\mathfrak{R}$ and represents γ on the part $|w| < M^*$ of the Riemann surface of $\log(w - M^*\zeta)$, never taking the value $M^*\zeta$. Then clearly

(2) $$F^*(z) = M^* F\left(\frac{G}{M^*}\right)$$

belongs to $\mathfrak{R}$ (since the argument ζ cannot occur on the right) and is subordinate in γ to $M^* F$. Thus the inequality $M < M^*$ of Theorem 224 cannot become an equality, and the theorem is best possible.

23.9. As an example, let $a > 1$, and consider again the function (cf. § 23.5)

$$F_a = \frac{z}{az+1}.$$

This has the pole $\zeta = -1/a$. Hence, by § 23.8 (1),

$$e^{-X} = \frac{1}{a}, \quad M^* = \frac{a^2-1}{2aX}.$$

With the notation of § 23.5 (2) we have further

$$M^*c = \frac{1}{2X}; \quad M^*r = \frac{1}{2aX} = \frac{M^*c}{a} = M^*c\,e^{-X} = M^*c\,e^{-1/(2M^*c)}.$$

The function $F_a^*(z)$ derived from F_a by the formula of § 23.8 (2) represents γ on the part $|w-M^*c| > M^*r$ of the Riemann surface of $\log(w-M^*c)$. Replacing M^*c by c, we obtain the following improvement of Theorem 219 (5) when f belongs to $\mathfrak{R}$ instead of $\mathfrak{M}$:

THEOREM 225.—*Let c be an arbitrary complex number $\neq 0$. Every function $f(z)$ of $\mathfrak{R}$ takes some value w with*

(1) $$|w-c| < |c|\,e^{-1/(2|c|)} = r_2^*(c)$$

That function F of $\mathfrak{R}$ must be excluded which maps γ on the part

$$|w-c| > r_2^*(c)$$

of the Riemann surface of $\log(w-c)$.

[A simple direct proof of this theorem is possible by using the function $F_2(z)$ of § 23.7 as superordinate function.]

We add here the corresponding improvements for the class $\mathfrak{R}$ of the results (iv) and (v) of § 23.6, omitting the elementary calculations.

(i) *Let $x > 0$, and let ϑ be real. Every function f of $\mathfrak{R}$ takes some value w on the stretch*

(2) $$\frac{1}{4x}\tanh x < |w| < \frac{1}{4x}\operatorname{cotanh} x \quad (\operatorname{arc} w = \vartheta).$$

That function of $\mathfrak{R}$ must be excluded which represents γ on the Riemann surface of $\log\bigl(w-1/(4x)\bigr)$ *slit along this stretch (in every sheet).*

(ii) *Let $0 < \rho < 1$. Every function $f(z)$ of $\mathfrak{R}$ takes some value w on each arc A of the circle $|w| = \rho$ with aperture*

(3) $$4 \operatorname{arc\,sin} e^{-x}, \quad \textit{where} \quad \frac{1-e^{-2x}}{2x} = \rho \quad (x > 0).$$

That function of $\mathfrak{R}$ must be excluded which represents γ on the Riemann surface of $\log(w-a)$, *slit along A (in each sheet), where a is the mid-point of A.*

23.10. We now enter on a digression, and must begin with the following

LEMMA.—*Let $w=f(z)$ belong to $\mathfrak{R}$ and satisfy $|f|\leqslant M$ in γ. Then the inverse function $z=\phi(w)$, with $\phi(0)=0$, is regular and schlicht for $|w|<1/(6M)$, and represents this circle conformally on a domain d inside $|z|<\frac{1}{2}/(1+M)$, while f is schlicht in d and represents it on the w-circle*†.

COROLLARY.—*Let $w=F(\zeta)=a_1\zeta+a_2\zeta^2+\ldots$, where $a_1\neq 0$, be regular and satisfy $|F|\leqslant M$ in $|\zeta|<R$. Then the inverse function $\Phi(w)$, with $\Phi(0)=0$, is regular and schlicht for*

$$|w|<S=\tfrac{1}{6}\frac{|a_1|^2 R^2}{M},$$

which circle is the image by $F(\zeta)$ of a domain in $|\zeta|<R$.

The corollary, an easy deduction from the main theorem, is what we actually use. To prove the theorem, let $f=\sum\limits_1^\infty a_n z^n$ in γ. Then $a_1=1$, $M\geqslant 1$, $|a_n|\leqslant M$. If $|z|=\rho<1$,

$$(1)\qquad |f(z)|\geqslant\rho\left(1-\sum_2^\infty|a_n|\rho^{n-1}\right)\geqslant\rho\left(1-\frac{M\rho}{1-\rho}\right).$$

Let $\rho^*=\dfrac{1}{2(1+M)}$. Then

$$(2)\qquad f(z)\neq 0\quad(0<|z|\leqslant\rho^*),$$

while on $|z|=\rho^*$

$$(3)\qquad |f(z)|\geqslant\rho^*\left(1-\frac{M\rho^*}{1-\rho^*}\right)=\frac{1}{2(1+2M)}\geqslant\frac{1}{6M}.$$

Since $f(0)=0$, $f'(0)=1$, it follows from (2) and (3) that f takes in $|z|<\rho^*$ each value w satisfying $|w|<1/(6M)$ exactly as often as the value $w=0$, that is exactly once. Hence $z=\phi(w)$ is regular and schlicht in $|w|<1/(6M)$, and maps it on a domain d inside $|z|<\rho^*$.

We are now in a position to prove

THEOREM 226 (Bloch).—*Let $w=f(z)$ belong to $\mathfrak{R}$. Then $\mathfrak{W}(f)$ contains an open schlicht circle of radius $\frac{1}{24}$; that is to say*‡ *there exists a simply-*

† The lemma gives a lower bound for the radius of convergence (about $w=0$) of the function $\phi(w)$ inverse to $f(z)$.

‡ To express the result in a form that does not mention Riemann surfaces.

connected domain D in γ such that f is schlicht in D and the f-image of D contains an open w-circle of radius $\frac{1}{24}$†.

This exceedingly odd and striking theorem resembles Theorem 222, but the condition $|f| < M$ of the latter theorem is completely dropped (and nothing replaces it, except that $\tau(M)$ becomes an absolute constant). It can be used to prove some of the "Picard" theorems of the next section (indeed it gives proofs in which the function theory involved is of the least possible "depth"). For these reasons this seems the best place for it, although it is not actually connected with subordination.

Given the *existence* of the theorem, and (what will become plausible in a moment) that the lemma is relevant to its proof, any competent analyst should be able to find one: it is true that the oddity of the theorem is reflected in the critical step of the proof, but the step is forced, and then not difficult to make.

We may suppose $f(z)$ regular in $|z| \leqslant 1$. If the domain D is a circle, as we naturally begin by trying, let it have centre z_0, with $|z_0| = \rho$. We want

$$w = F(\zeta) = f(z_0+\zeta) - f(z_0) = a_1\zeta + a_2\zeta^2 + \ldots,$$

where $$a_1 = f'(z_0),$$

to have an inverse $\Phi(w)$, schlicht in $|w| \leqslant s$, where s is at least some absolute constant (the value of which happens to come to $\frac{1}{24}$). The corollary of the lemma provides a permissible s given by

$$6s = \frac{|a_1|^2 R^2}{M},$$

where R is some radius, necessarily less than or equal to $1-\rho$, in which F is regular and $|F| \leqslant M = M(R)$. Now given z_0 and R,

$$M = \operatorname*{Max}_{|\zeta| \leqslant R} |f(z_0+\zeta) - f(z_0)| = \operatorname*{Max}_{(\phi)} \left| \int_0^R f'(z_0 + re^{i\phi})\, dr \right|.$$

A crude upper bound for this is

$$M \leqslant \int_0^R \mu(\rho+R)\, dr = R\mu(\rho+R),$$

where $$\mu(t) = \operatorname*{Max}_{|z| \leqslant t} |f'(z)|;$$

† The constant $\frac{1}{24}$ is not best possible; the true value is not known.

$\mu(t)$ increases from 1 at $t=0$ to a finite value at $t=1$. The corresponding value of s is now given by

$$6s = \frac{|f'(z_0)|^2 R}{\mu(\rho+R)},$$

and we want to make this as large as possible, and greater than an absolute constant. We have at our disposal ρ, R ($\leqslant 1-\rho$), and $\arg z_0$. The last we naturally choose to make $|f'(z_0)|$ have its maximum value $\mu(\rho)$; then

$$6s = \frac{R\mu^2(\rho)}{\mu(R+\rho)}.$$

The denominator increases with R (for given ρ), and $\rho+R$ must certainly not be 1 [$\mu(1)$ can be arbitrarily large]; it is reasonable to try $R = \frac{1}{2}(1-\rho)$, which makes the ratio of the distances of ρ and $\rho+R$ from 1 equal to 2†. We thus have a permissible s given by

$$12s = \frac{(1-\rho)\,\mu^2(\rho)}{\mu\{\frac{1}{2}(1+\rho)\}}.$$

We still have ρ at our disposal, and the final question is: is there an absolute constant a (ultimately $a=1$) such that, given any

$$f'(z) = 1+b_1 z+\dots$$

regular in $|z| \leqslant 1$, there is a ρ in $0 \leqslant \rho < 1$ such that

$$\mu\{\tfrac{1}{2}(1+\rho)\} \leqslant 2a(1-\rho)\,\mu^2(\rho):$$

or, simplifying by $1-\rho = x$ and multiplication by x, is there an x in $0 < x \leqslant 1$ such that

$$\tfrac{1}{2}x\mu(1-\tfrac{1}{2}x) \leqslant a\{x\mu(1-x)\}^2\,? \tag{4}$$

The answer is affirmative, and it is sufficient to know about $\mu(x)$ that it is continuous in $0 \leqslant x \leqslant 1$, and $\mu(0) = 1$. The graph of

$$y = h(x) = x\mu(1-x)$$

in $0 \leqslant x \leqslant 1$ is continuous, starts at 0 at $x=0$, and is 1 at $x=1$. Let ξ be the least value such that $h(\xi)=1$; ξ cannot be 0 (but may be 1). Then $h(\frac{1}{2}\xi) < 1 = h^2(\xi)$, and ξ gives a suitable value of x when a is 1.

By inverting the point of view we have the following result, which is what we shall actually apply to prove "Picard" theorems.

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† Any absolute ratio greater than 1 would give some final absolute constant.

COROLLARY.—*Suppose* $g(z) = b_0+b_1z+b_2z^2+\ldots$ *is regular in* γ *and has some missing value* w_0 *in every circle* $|w-c| < k$ *of given radius* k. *Then*

(5) $$|b_1| < 24k.$$

If $b_1 \neq 0$, we apply Bloch's theorem to $f = (g-b_0)/b_1$. There exists an open circle of radius $\frac{1}{24}$ all of whose points are values taken by f. Hence there is a circle of radius $\frac{1}{24}|b_1|$ all of whose points are values taken by g. This radius must be smaller than k.

24, 25, 26. *The functions P, Q, R.*

24.1. We denote by $P(z)$ any ("Picard") function which is regular and never takes the values 0 or 1 in γ; by $Q(z)$ any function which is regular and never takes any value $\pm 2n\pi i$ ($n = 0, 1, 2, \ldots$) in γ; by $R(z)$ any function which is regular and never takes any value $-4\pi^2 n^2$ in γ. It is evident that the necessary and sufficient condition for a function to be a Q is that it should be of the form $\log P$, and that the necessary and sufficient condition for a function to be an R is that it should be of the form Q^2.

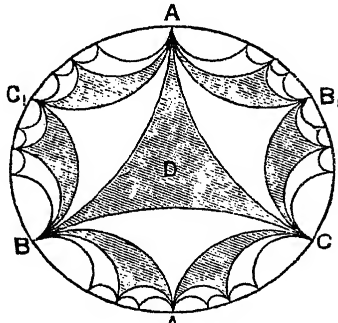

Fig. 9.

When P and $\bar{P}$ occur together, it is to be understood that $\bar{P}$ is also a function $P(z)$, and that P is subordinate to $\bar{P}$; similarly when Q, $\bar{Q}$ or R, $\bar{R}$ occur together. Finally we denote the coefficient of z^n in P by p_n, in Q by q_n, and in R by r_n.

We use the symbol $A(x, y, \ldots)$ to denote a positive constant depending only on the parameters shown explicitly, viz. $x, y, \ldots$, but not the same at different occurrences. In particular A always denotes a positive absolute constant, but not always the same one. When we wish to preserve the identity of an A we write $A_1, A_2, \ldots$.

24.2. In Fig. 9 A, B, C are the points $z = i$, $e^{-\frac{5}{6}\pi i}$, $e^{-\frac{1}{6}\pi i}$, and AB, BC, CA are circular arcs orthogonal to the circle $|z| = 1$. (ABC is thus an equilateral curvilinear triangle with zero angles.) Let $w = \mu(z)$ represent D, the interior of triangle ABC, on Π, the upper half w-plane, the points A, B, C of the z-plane corresponding to $w = 0$, 1, ∞ of the w-plane.

Invert the triangle ABC in each (primary) arc AB, BC, CA of the triangle. Now invert the whole resulting figure in every secondary arc then the whole resulting figure in every tertiary (*i.e.* outermost) arc, and so on. Inversion in a circle converts a figure exterior to the circle into one interior to it, preserves angles, and leaves an orthogonal circle invariant. Hence we obtain a figure composed of sides of zero-angled triangles, orthogonal to the circle ABC, and with all corners on this circle. It is plausible that the complete figure inverts into itself with respect to *any* side (shaded and unshaded regions being, however, interchanged): this property is actually true, and follows without difficulty from the geometrical principle: "*if γ, δ are circular arcs inverse with respect to a circle a, then the circles γ', δ', a' obtained by inversion in any fourth circle have the same property*".

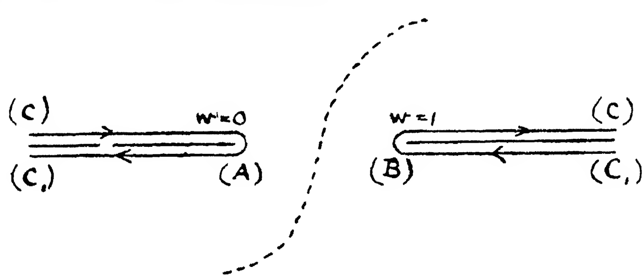

Fig. 10.

We observe next that the greatest of the outermost arcs of the n-th stage is small when n is large. If not, there must exist an arc β (orthogonal to ABC) whose interior is free of sides of the figure, for sides do not intersect. It is, however, evident that there must exist a contracting sequence of sides, one for each stage of the generating process, spanning β; these have a limiting arc β_0 spanning or identical with β, the interior of β_0 being free of sides. But evidently the inverse of triangle ABC in a side of the figure nearly identical with β_0 must enter the interior of β_0.

Next, every point P of the circumference $|z| = 1$ that is not one of the denumerable set of corners is spanned by an infinity of sides β_n contracting on to P as a limit point. Since sides do not intersect, the complete triangles of which β_{n+1} is a side are inside β_n.

From this we conclude finally that every point in $|z| < 1$ is ultimately covered by a triangle and that every point (corner or not) of the circumference $|z| = 1$ has an infinity of arbitrarily small triangles near it.

We show now that *the function $\mu(z)$ can be continued throughout $|z| < 1$, and has $|z| = 1$ for a line of essential singularities.* By the symmetry (or inversion) principle (Theorem 118) μ is regular across AB and takes in ABC_1, at a point inverse with respect to AB to a z of D, the value conjugate to $\mu(z)$. [Then $w = \mu(z)$ represents the interior of AC_1BC on the domain bounded by the continuous lines of Fig. 10.] Similarly for a path crossing any number of sides. μ is "schlicht" in every shaded, and in every unshaded region, taking in shaded regions values w belonging to Π, in unshaded ones values belonging to $\bar{\Pi}$, the lower half w-plane. A point of the circumference $|z| = 1$ has an infinity of small triangles near it; hence μ takes any value other than 0 or 1 infinitely often near the point. We see also, since μ is "schlicht" in each quadrilateral composed of two adjacent triangles that $\mu' \neq 0$ in $|z| < 1$.

It follows that $\phi(w)$, the many-valued function inverse to μ, *has $w = 0$, 1 (and ∞) as its only singularities.* If, for example, w_0 is in Π, there exists in each shaded triangle a z such that $\mu(z) = w_0$, and there exists a branch of $\phi(w)$, regular in Π and so in the neighbourhood of w_0, such that $\phi(w_0) = z$. There will exist an infinity of branches of ϕ, but each is regular at w_0.

24.3. *The Riemann surface $\mathfrak{W}$ of $w = \mu(z)$.*

To a z-path from a point of triangle ABC to one of ABC_1 corresponds a w-path represented by the broken line in Fig. 10. We take an infinite number of half-planes Π and $\bar{\Pi}$. Start with Π_0, a particular Π, corresponding by $w = \mu(z)$ to D. Corresponding to AC_1B we must have a $\bar{\Pi}_1$ joined to Π across (0, 1). If we now (in the z-circle) cross BC_1 into A_2C_1B we must join a Π_2 to $\bar{\Pi}_1$ across $(1, \infty)$. And so on. We obtain a surface $\mathfrak{W}$ of ∞^2 sheets with winding points of infinite order, in each sheet, at $w = 0, 1, \infty$. *Note that the w-paths corresponding to two z-paths from P to Q lead to the same point of $\mathfrak{W}$.* [Proof by induction: if a path is deformed to go into one new triangle it comes out by the same arc that it went in by.]

The equation $w = \mu(z)$ represents γ on $\mathfrak{W}$. The inverse function $\phi(w)$ is regular and "schlicht" on $\mathfrak{W}$ (the points $w = 0$, 1, ∞ in every sheet being taken as boundary points).

The precise definition of $\mathfrak{W}$ is as follows. Take an infinite number of planes, each cut from $-\infty$ to 0 and from 1 to $+\infty$. Take one such

plane, W_1. This has four edges, two companion edges from $-\infty$ to 0, two companions from 1 to $+\infty$. To each edge we join the companion edge belonging to another cut plane (one plane for each edge, four new planes in all). The assemblage W_2 of three sheets has twelve free edges (and corresponds to a dodecagon in the z-figure). We now deal with these free edges in a similar manner, affixing the companion edge of a new cut plane to each. And so on. We obtain a sequence W_1, W_2, ... of surfaces, and $\mathfrak{W}$ is $\lim W_n = \Sigma W_n$.

The following remarks may instruct the suitably informed reader. They are not used in the sequel and may be ignored if necessary.

(1) $\mathfrak{W}$ is simply connected. This follows from the one-one continuous correspondence with γ; but it is intuitive from first principles, since a closed contour on $\mathfrak{W}$ cannot surround a branch point and can shrink continuously to evanescence.

(2) The existence of a function $w = F(z, p_0)$ representing γ on $\mathfrak{W}$ with $F(0) = p_0$ is a particular case of a general theorem that γ can be represented (with a triple arbitrariness) on any simply-connected surface of a certain type (to which $\mathfrak{W}$ conforms). (It then follows by an argument given below that $P(z)$ is subordinate to F. Actually we shall arrive at the function F *via* the μ of the preceding argument.)

(3) [Sketch only.] $F(z)$ is certainly an automorphic function. For let P, P' be any two homologous points (points with the same w) of $\mathfrak{W}$. To a path PQ from P to a variable Q corresponds a homologous path $P'Q'$, and the Q' corresponding to a given Q does not depend on the path PQ. The relation set up between Q and Q' is evidently conformal in the geometrical sense (is one-one and preserves angles), and transforms $\mathfrak{W}$ into itself. If z, z' are the z-points corresponding to Q, Q', we may expect the transformation from z to z' to be a conformal transformation of γ into itself, and this is in fact the case. Such a transformation is necessarily linear (by Theorem 119), and since w is the same for Q, Q', it leaves $F(z)$ unaltered. $F(z)$ is therefore invariant for *some* linear transformations, and therefore for their group.

24.4. PICARD'S THEOREM FOR INTEGRAL FUNCTIONS.—*An integral function $F(z)$ which takes neither of two distinct values a, b is a constant.*

We may suppose $a = 0$, $b = 1$. Then $g(z) = \phi(F)$ is regular for all z. But $|g| < 1$. By Liouville's theorem $g =$ constant, $F = \mu(g) =$ constant.

COROLLARY.—*A function $F(z)$, meromorphic in the whole plane and having three distinct missing values a, b, c, is a constant.*

$(F-a)^{-1}$ is an integral function with $(b-a)^{-1}$ and $(c-a)^{-1}$ as missing values.

24.5. PICARD'S THEOREM FOR A CIRCLE (Schottky).—

$$M(\rho, P) < A(\rho, p_0).$$

We have $p_0 = P(0) \neq 0, 1$. Hence there exists a unique $z = a_0$, corresponding by $w = \mu(z)$ to $w = p_0$ and lying in $D + D_A + (BA) + (AC)$ (D_A being the inverse of D with respect to BC). Then

$$\zeta = \frac{z - a_0}{a_0' z - 1}, \quad \text{or} \quad z = \frac{\zeta - a_0}{a_0' \zeta - 1},$$

transforms $\gamma(z)$ into $\gamma(\zeta)$, $z = a_0$ into $\zeta = 0$, and the triangle ABC into a ζ-triangle $A'B'C'$ with zero angles. The function

$$w = F(\zeta, p_0) = \mu(z) = \mu\left(\frac{\zeta - a_0}{a_0'\zeta - 1}\right),$$

uniquely defined for given p_0, represents $\gamma(\zeta)$ on $\mathfrak{W}$, a $w = p_0$ (point Q say) corresponding to $\zeta = 0$. $\mathfrak{W}$ is locally schlicht, and every sheet contains all values w except $0, 1, \infty$. *A $P(\zeta)$ must have its representative point, continued anaytically from Q ($w = p_0$) corresponding to $\zeta = 0$, confined to $\mathfrak{W}$, and so, by* §22.1, *must be subordinate to $F(\zeta)$.* [Compare §23.7.] Consequently

$$M(\rho, P) \leqslant M(\rho, F) = A(\rho, p_0).$$

24.6. There is an extension of the last result.

THEOREM 227.—*If $|P(0)| < \varpi$, then*

$$M(\rho, P) < A_1(\rho, \varpi).$$

COROLLARY.—*If $|P(0)| < \varpi$, then $|p_1| < A_1(\varpi)$.*

Suppose first that p_0 is confined to the closed D' defined by $|w| \leqslant \varpi$, $|w| \geqslant \frac{1}{2}$, $|w - 1| \geqslant \frac{1}{2}$. Evidently a_0 is a continuous function of p_0 (if we allow a_0 to go out of the fundamental region) and so $F(\zeta, p_0)$ is a continuous function of the pair of variables $\arg \zeta$ and p_0 (if $|\zeta| = \rho < 1$ is kept fixed and p_0 is confined to D'). Hence $M(\rho, F)$ is continuous in p_0,

$$M(\rho, F) < A(\rho, \varpi),$$

$$(1) \qquad |P| < A(\rho, \varpi) \qquad (|\zeta| = \rho).$$

We show next that we may add the circle $|w-1| \leqslant \frac{1}{2}$ to D' (giving D_1') without prejudice to the form of this inequality. The two branches of $\sqrt{P}$ are P's, and their constant terms are $p_0^{\frac{1}{2}}$ and $-p_0^{\frac{1}{2}}$. If now $|p_0-1| \leqslant \frac{1}{2}$, *one* of these must lie in D'. By (1) we have

$$|\sqrt{P}| < A(\rho, \varpi), \quad |P| < A(\rho, \varpi).$$

Finally, we may similarly add $|w| \leqslant \frac{1}{2}$ to D_1', since if $|p_0| \leqslant \frac{1}{2}$, $1-P$ is a P whose p_0 belongs to D_1', $|1-P| < A(\rho, \varpi)$ and $|P| < A(\rho, \varpi)$. But D_1' has now become $|w| \leqslant \varpi$, and we have proved the main theorem.
The corollary follows since $\frac{1}{2}|p_1| \leqslant M(\frac{1}{2}, \varpi) < A(\varpi)$.

24.71. The following theorem is an application of Theorem 227.

Suppose that $\alpha > \beta$, and that $\psi(z)$, with only a finite number of singularities in $x > \beta$, has the properties; (1) *for every $\delta > 0$*

$$\Psi(x) = \overline{\lim_{y\to\infty}}\, |\psi(x+iy)|$$

is bounded in $x \geqslant \alpha+\delta$; (2) *for every $\delta > 0$ Ψ is not bounded in $x > \alpha-\delta$. Then for every $\delta > 0$ ψ takes every value, with one possible exception, an infinity of times in $|x-\alpha| < \delta$, $y > 0$.*

Suppose this false. Then there exist a, b, $\delta > 0$, η, such that $\psi \neq a, b$ in $|x-\alpha| \leqslant 5\delta$, $y > \eta$; in particular $\psi \neq a, b$ in $|z-z_0| < 4\delta$ when $x_0 = \alpha+\delta$ and y_0 is large. We may suppose $a = 0$, $b = 1$ [otherwise let $\Psi = (\psi-a)/(b-a)$]. Since $|\psi(z_0)| < K$, Theorem 227 gives

$$\operatorname*{Max}_{\leqslant \frac{1}{2}} |\psi(z_0+4\delta\rho e^{i\theta})| < A_1(K, \tfrac{1}{2}) = K_1;$$

$|\psi| < K_1$ in $|z-z_0| < 2\delta$.
Since the z circle touches $x = \alpha-\delta$ and y_0 may run through all large values it follows that ψ is bounded in $x > \alpha-\delta$ as $y \to +\infty$, and this is false.

24.72. To give point to this application we digress to prove the following famous result (Bohr).

The function $\zeta(z) = \Sigma n^{-z}$ has the property of ψ, with $\alpha = 1$.

It is well known that $\zeta(z)$ is regular except for $z = 1$. In $x \geqslant 1+\delta$, $|\zeta| \leqslant \Sigma n^{-1-\delta} < K$: it remains only to show that ζ is not bounded in any

$x > 1-\delta$, $y > 1$, and we shall actually prove that ζ is not bounded in $x > 1$, $y > 1$. To this end we must prove first:

DIRICHLET'S THEOREM.—*Given* $\theta_1, \theta_2, \ldots, \theta_N$, $\tau > 0$, *and a positive integer* q, *there exists a* $t \geqslant \tau$, *such that*

$$|\{t\theta_r\}| \leqslant 1/q \quad (r \leqslant N)\dagger,$$

and in fact such a t *can be found satisfying also* $t \leqslant \tau q^N$.

COROLLARY. *If* $\sum_1^\infty a_n$ *is a convergent series of non-negative terms, then*

$$\overline{\lim_{y\to\infty}} \mathfrak{R} \sum_1^\infty a_n e^{iy\theta_n} = \sum_1^\infty a_n.$$

Consider the unit cube in N dimensions: divide each side into q equal parts and draw, through the points of division, "planes" parallel to the coordinate planes, thus dividing the cube into q^N compartments. Consider the q^N+1 points

$$(\tau\nu\theta_1,\ \tau\nu\theta_2,\ \ldots,\ \tau\nu\theta_N) \quad (\nu = 1, 2, \ldots, q^N+1)$$

reduced (mod 1) in each coordinate. All lie in the cube, two therefore, say for ν_1 and ν_2, lie in one compartment. Then

$$|\{\tau(\nu_1-\nu_2)\,\theta_r\}| \leqslant 1/q \quad (r \leqslant N),$$

and we have what we want by taking $t = \tau|\nu_1-\nu_2| \geqslant \tau$.

For the corollary, the theorem gives at once

$$\overline{\lim} \mathfrak{R} \sum_1^N a_n e^{iy\theta_n} = \sum_1^N a_n,$$

and we can replace N by ∞ on account of the uniform convergence.

We return now to $\zeta(z)$. Given a (large) positive g we choose first x, satisfying $1 < x < 2$ and such that

$$\Sigma n^{-x} > 6g;$$

then N so that
$$\sum_{N+1}^\infty n^{-x} < g;$$

† $\{x\}$ denotes in general the difference between x and the integer nearest to x, and in the ambiguous case $x = r+\frac{1}{2}$ it denotes $+\frac{1}{2}$.

and then y greater than an arbitrary η and such that

$$\cos(y\log n) \geqslant \tfrac{1}{2} \quad (n \leqslant N).$$

Then
$$\Re\zeta(x+iy) = \sum^{N} n^{-x}\cos(y\log n) + \sum_{N+1}^{\infty}$$

$$\geqslant \tfrac{1}{2}\sum^{N} n^{-x} - \sum_{N+1}^{\infty} n^{-x} > \tfrac{1}{2}\sum^{\infty} n^{-x} - 2\sum_{N+1}^{\infty} n^{-x}$$

$$\geqslant 3g - 2g = g.$$

Since this happens for some y greater than an arbitrary η, and g is arbitrary, it follows that ζ is not bounded in $x > 1$, $y > 1$.

Taking as known, (i) the corollary of the lemma, (ii) the general principle

$$\overline{\lim} \geqslant \overline{\lim}\ \overline{\lim},$$

valid for any real function of x, y (defined in the relevant range), we can condense the above proof into the single line:

$$\overline{\lim_{y\to\infty\ (x>1)}} \Re\zeta(z) \geqslant \overline{\lim_{x\to 1+0}}\ \overline{\lim_{y\to\infty}} \Re\sum_{1}^{\infty} n^{-x-iy} = \overline{\lim_{x\to 1+0}} \sum_{1}^{\infty} n^{-x} = \infty.$$

24.73. Theorem 227 gives an easy proof that a uniform function with 3 missing values in the neighbourhood of an essential singularity is necessarily a constant†. We may suppose the essential singularity at $z = \infty$, and the missing values to be 0, 1, ∞. It is enough, by the maximum modulus principle, to prove that $F(z)$, regular and never 0 or 1 in $R_0 < |z| < \infty$, is uniformly bounded on *some* arbitrarily large circles $|z| = R$. Now by Weierstrass's Theorem there exist large circles $|z| = R$ each containing a point, z_0 say, with $|F(z_0)| < 1$ (say). We can find a chain of $N+1 < A$ circles $C_0, C_1, \ldots, C_N$, with centres $z_0, z_1, \ldots, z_N$ on $|z| = R$ and satisfying $\operatorname{Max}(|z_1 - z_0|, |z_2 - z_1|, \ldots, |z_0 - z_N|) < \tfrac{1}{4}R$, while the circles themselves have radius $\tfrac{1}{2}R$. Applying Theorem 227, with $\rho = \tfrac{1}{2}$, to $C_0, C_1, \ldots, C_N$ successively, we find $|F(z_n)| < A$ $(0 \leqslant n < N)$; by a final application to each C_n we have $|F(z)| < A$ in each circle $|z - z_n| < \tfrac{1}{4}R$, and consequently on the whole circumference $|z| = R$.

† In the early history of the subject this natural extension of Picard's original theorem was regarded as a new major problem. That this is no longer true is another instance of the effectiveness of the "ϖ" form (Theorem 227) as compared with the "p_0" one.

24.8. The inequalities so far proved for P involve functions $A(\rho, p_0)$ or $A(\rho, \varpi)$ of unknown form, depending on the form of $F(z, p_0)$. We shall show now that without any further inquiry into the special nature of F, and assuming merely its existence, we can obtain results of fair precision. We prove first:

LEMMA.— $$|q_1| < A(|q_0|+1).$$

Let n be the least integer such that $(|q_0|+1)/(2n\pi) < \frac{1}{2}$. Then $Q(z)/(2n\pi i) = a_0+a_1 z+\ldots$ is a P, and $|a_0| = |q_0|/(2n\pi) < \frac{1}{2}$. By Theorem 227, Cor., we have $|a_1| < A$, $|q_1| < An < A(|q_0|+1)$, the result of the lemma.

Our subsequent arguments can take either of two forms.

(i) Consider $\psi(z) = Q(\zeta) = b_0+b_1 z+\ldots$, where $\zeta = (z-z_0)/(z_0' z-1)$ and $|z_0| = \rho < 1$. ψ is a $Q(z)$, and its q_0, q_1 are

$$Q(z_0) \quad \text{and} \quad -Q'(z_0)(1-|z_0|^2).$$

Hence
$$|Q'(z_0)| < A\frac{|Q(z_0)|+1}{1-|z_0|^2}.$$

$$\frac{d}{d\rho}\log\{1+|Q(z_0)|\} \leqslant \frac{|Q'(z_0)|}{1+|Q(z_0)|} \leqslant \frac{A}{1-\rho^2},$$

$$\log\{1+|Q(z_0)|\} < A\log\frac{2}{1-\rho}+\log(1+|q_0|),$$

$$|Q(z_0)| < 1+|Q(z_0)| < A(1+|q_0|)(1-\rho)^{-A}.$$

Thus the "order" of $Q(z)$ in the circle does not exceed an A.

(ii) In proving the last inequality we may suppose without loss of generality that $z_0 = \rho$. Let $\chi(z) = Q\{\rho+(1-\rho)z\}$: χ is a $Q(z)$, and its q_0, q_1 are $Q(\rho)$. $(1-\rho)\,Q'(\rho)$. The lemma gives

$$(1-\rho)|Q'(\rho)| < A\{1+|Q(\rho)|\},$$

and the argument is completed on the same lines as before.

We shall meet again with arguments on the lines of (i) and (ii); we call them respectively the ζ-argument and the ρ-argument. The former gives a better inequality [since $1-\rho^2 \sim 2(1-\rho)$ as $\rho \to 1$] and in some connexions gives a best possible one. The latter, however, is always

simpler in detail, and therefore preferable where neither method is best possible.

What these arguments do is to show that in suitable circumstances [those in which $w = f(z)$ is restricted to move on a *given* Riemann surface] all points of γ are roughly on an equal footing : we can transfer a property of the origin to an arbitrary point z_0. When the property of the origin is of type $|a_n| < F(a_0, a_1, \ldots, a_{n-1})$ the transferred property is a differential inequation for $f(z)$ of the n-th order.

24.9. The lemma can be deduced from Bloch's Theorem (Theorem 226) [and so without assuming even the existence of the function $\mu(z)$].

Let $f = \sqrt{(Q/2\pi i)}$ (any determination). f is regular in γ, and $f \neq \pm\sqrt{n}$ $(n \geqslant 0)$. Since the function arc cosh w has $w = \pm 1$ as its only singularities† (any determination of) the function

$$g = \operatorname{arc\,cosh} f$$

is regular in γ, and $g \neq \pm \operatorname{arc\,cosh} \sqrt{n} \pm 2m\pi i$ (m an integer)‡.

Now $\operatorname{arc\,cosh}\sqrt{(n+1)} - \operatorname{arc\,cosh}\sqrt{n} \to 0$ as $n \to \infty$. Hence there is an absolute constant k such that any w-circle of radius k contains a missing value of g. By the corollary to Theorem 226 we obtain

$$|g'(0)| < 24k = A.$$

Now
$$g' = \frac{f'}{\sqrt{(f^2-1)}} = \frac{1}{4\pi i}\frac{Q'}{\sqrt{\left\{\frac{Q}{2\pi i}\left(\frac{Q}{2\pi i}-1\right)\right\}}},$$

and so

$$|q_1| < A\sqrt{\left\{\frac{|q_0|}{2\pi}\left(\frac{|q_0|}{2\pi}+1\right)\right\}} < A(|q_0|+1),$$

the desired result.

25.1. So far we have used only the bare existence of the function $\mu(z)$. To obtain "best possible" results we need full information about it; $\mu(z)$ is actually an elliptic modular function.

Let $z = S(\tau)$ be the linear transformation that makes the points $e^{-\frac{1}{3}\pi i}$ (points A, B, C of figure 10) correspond to the points

† Consider the derivative.

‡ Otherwise $\cosh g = f = \cdot$

$\tau=\infty$, 0, 1, respectively (points A, B, C of figure 11). Then $\mu(z)$ becomes the function

$$\lambda(\tau)=\mu\big(S(\tau)\big). \tag{1}$$

$\lambda(\tau)$ transforms the upper half τ-plane into our Riemann surface $\mathfrak{W}$, the "circular" τ-triangle A, B, C being transformed into an upper half λ-plane, the points $\tau=\infty$, 0, 1 corresponding respectively to the points

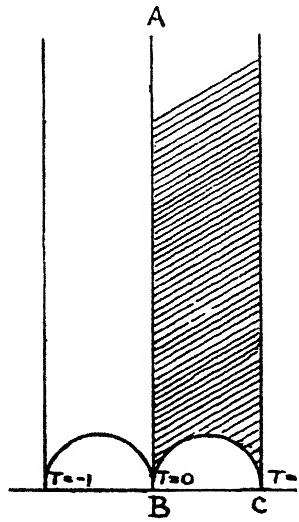

Fig. 11.

$\lambda=0$, 1, ∞. Now it is well known that the elliptic modular function known as $k^2(\tau)$ transforms our τ-triangle in exactly this manner. It follows that

$$\lambda(\tau)=k^2(\tau). \tag{2}$$

This then is the nature of our function $\mu(z)$.

Consider now the function

$$\mu^*(\zeta)=2\big(\lambda(\tau)-\tfrac{1}{2}\big) \quad \left(\tau=i\,\frac{1-\zeta}{1+\zeta}\right). \tag{3}$$

It transforms the unit circle $\gamma(\zeta)$ into the Riemann surface $\mathfrak{W}^*=2(\mathfrak{W}-\frac{1}{2})$, which is of exactly the same nature as the surface $\mathfrak{W}$, except that its winding points are at $w^*=\infty$, 1, -1, which are missing values of $\mu^*(\zeta)$ in γ. The "circular" ζ-triangle with corners -1, 1, i, and sides orthogonal to $|\zeta|=1$, is transformed into an upper half w^*-plane, the corners

corresponding to $w^* = -1, 1, \infty$ respectively. The symmetry principle shows that the function $-\mu^*(-\zeta)$ transforms this triangle in exactly the same manner. Hence $\mu^*(\zeta) = -\mu^*(-\zeta)$, *i.e.* $\mu^*(\zeta)$ is an odd function. In particular $\mu^*(0) = 0$, and so $\lambda(i) = k^2(i) = \frac{1}{2}$. Clearly $\mu^{*\prime}(0) > 0$. Hence

$$\mu^{*\prime}(0) = -4i\lambda'(i) = 4|\lambda'(i)| = a^{-1},$$

where the value a is known to be $4\pi^2/\Gamma^4(\frac{1}{4})$. Finally the odd function

$$\mu_1(z) = a\mu^*(z)$$

belongs to $\mathfrak{R}$ (see § 23), has $\pm a$ as missing values in γ, and transforms γ into the Riemann surface $a\mathfrak{W}^*$ with winding points at $\pm a$ (and ∞).

THEOREM 228.—*Every function $w = f(z)$ of $\mathfrak{R}$ takes at least one value of each couple $\pm w$ belonging to the circle*

$$(4) \qquad |w| \leqslant a = \frac{4\pi^2}{\Gamma^4(\frac{1}{4})} \sim 0{\cdot}228\ldots.$$

In particular, every odd function of $\mathfrak{R}$ takes all values of this circle. If $w = \eta a$, $|\eta| = 1$, the (odd) function $\eta\mu_1(\eta^{-1}z)$ is excepted, and (4) *is best possible.*

Let $0 < w_0 \leqslant a$, say. If $f \neq \pm w_0$ in γ, f is subordinate to $t\mu_1(z)$, where $t = w_0/a \leqslant 1$. By Theorem 217 this is possible only if $t = 1$, that is $w_0 = a$, and then $f \equiv \mu_1$.

25.2. Put $z = e^{i\pi\tau}$ and consider the function

$$(1) \qquad H(z) = \lambda(\tau) = \lambda\left(\frac{\log z}{\pi i}\right).$$

If $\mathfrak{I}(\tau) > 0$, then $|z| < 1$. By the symmetry principle $\lambda(\tau)$ has the period 2, and so has $e^{i\pi\tau}$. Hence $H(z)$ is regular and uniform in $0 < |z| < 1$. Also $H \neq 0, 1$ for these z. On the other hand, $\mathfrak{I}(\tau) \to \infty$ and so $\lambda(\tau) \to 0$ as $z \to 0$. Hence $H(z)$ is regular at $z = 0$ also, and $H(0) = 0$. H transforms γ into a Riemann surface with an infinity of sheets and winding points at 0, 1, ∞, and differing from $\mathfrak{W}$ only by the fact that in *one* sheet the point $w = 0 = H(0)$ is not a winding point.

By a known formula for $\lambda(\tau)$ (given in § 26.6 below) we have

$$(2) \qquad H(z) = 16z \exp\left\{8 \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \frac{z^m}{1+z^m}\right\},$$

whence $H'(0) = 16$. The function $H_1(z) = 16^{-1}H(z)$ thus belongs to $\mathfrak{R}$, and applying Theorem 217 once more, we obtain

THEOREM 229 (Hurwitz). — *Every function $f(z)$ of $\mathfrak{R}$, vanishing at $z = 0$ only, takes all values $|w| \leqslant \frac{1}{16}$. If $w = \frac{1}{16}\eta$, $|\eta| = 1$, however, the function $\eta H_1(\eta^{-1}z)$ is excepted.*

26 . 1. As we have seen already in § 25 . 1, the elliptic modular function $\lambda(\tau) = k^2(\tau)$ transforms the upper half τ-plane into the Riemann surface $\mathfrak{W}$ in such a way that the shaded triangle A, B, C of figure 11 becomes an upper half λ-plane, the corners corresponding to $\lambda = 0,\ 1,\ \infty$ respectively.

By the symmetry principle it follows that, given $p_0 \neq 0, 1$, there exists in the region R composed of the shaded domain, its image in AB, and the (open) arcs (AB), (BC), a τ_0 such that $\lambda(\tau_0) = p_0$. Let τ_0' be the conjugate of τ_0, and

$$z = \frac{\tau - \tau_0}{\tau - \tau_0'} \quad \text{or} \quad \tau = \frac{\tau_0 - \tau_0' z}{1 - z},$$

so that the τ half-plane corresponds to γ. Let now $\lambda(\tau) = \bar{P}(z, p_0)$. This defines a unique $\bar{P}$ (given p_0). Then $w = \bar{P}(z)$ represents γ on $\mathfrak{W}$, and P is subordinate to $\bar{P}$. We have, then,

THEOREM 230.—*$P(z)$ is subordinate in γ to the function*

$$\bar{P}(z) = \bar{P}(z, p_0),$$

uniquely determined, for given p_0, by the relations

$$\bar{P}(z) = \lambda(\tau), \qquad z = \frac{\tau - \tau_0}{\tau - \tau_0'},$$

$$\lambda(\tau_0) = p_0,$$

(1) $$-1 < \mathfrak{R}\tau_0 \leqslant 1, \qquad |\tau_0 - \tfrac{1}{2}| \geqslant \tfrac{1}{2}, \qquad |\tau_0 + \tfrac{1}{2}| > \tfrac{1}{2};$$

where $\lambda(\tau)$ [or $k^2(\tau)$] is the elliptic modular function, and τ_0' is the conjugate of τ_0.

THEOREM 231.—(i) *$Q(z)$ is subordinate in γ to the function*

$$\bar{Q}(z) = \bar{Q}(z, q_0),$$

$$\bar{Q}(z) = \log \lambda(\tau), \qquad z = \frac{\tau - \tau_0}{\tau - \tau_0'},$$

(2) $$\log \lambda(\tau_0) = q_0,$$

where τ_0 *is subject to* (1), *and the determination of the logarithm is fixed by* (2).

(ii) $R(z)$ *is subordinate in* γ *to*

$$\bar{R}(z) = \bar{R}(z, r_0) = \{\bar{Q}(z, \sqrt{r_0})\}^2.$$

For if $p_0 = e^{q_0}$ we have, with appropriate logarithms throughout, $\bar{Q} = \log \bar{P}$ and $\bar{Q}(0) = q_0$. By Lemma 1, § 22, $Q = \log P$ is subordinate to $\bar{Q} = \log \bar{P}$. This proves the first part; the second is very similar.

26.2. Turning now to the study of the functions $\bar{P}$, $\bar{Q}$, $\bar{R}$, we have first

THEOREM 232.—

$$|\bar{Q}(z, q_0)| \leqslant \frac{A(q_0)}{1-|z|},$$

$$|\bar{P}(z, p_0)| \leqslant \exp\left(\frac{A(p_0)}{1-|z|}\right),$$

$$|\bar{R}(z, r_0)| \leqslant \frac{A(r_0)}{(1-|z|)^2}.$$

We postpone a proof of the first result; the second and third follow at once, since $\bar{P} = \exp \bar{Q}$, $\bar{R} = \bar{Q}^2$. We deduce at once from Theorem 232.

THEOREM 233.—

$$M(\rho, Q) \leqslant \frac{A(q_0)}{1-\rho},$$

$$M(\rho, P) \leqslant \exp\left(\frac{A(p_0)}{1-\rho}\right),$$

$$M(\rho, R) \leqslant \frac{A(r_0)}{(1-\rho)^2}.$$

There is little more of interest to be said about functions P, as such, and we shall be chiefly concerned with functions $Q\,(=\log P)$ and R.

26.3. Consider first the functions Q, and compare them with their subclass B, discussed in § 23.4. Whereas functions B have a set of "missing values" filling the whole straight line $\mathfrak{R}w = 0$, the assigned missing values of Q are only the discrete set $\pm 2n\pi i$; the B's are a very special sub-class of the Q's. None the less Theorem 233 shows that an inequality similar to § 23.4.(3) holds also for Q. It is natural to inquire whether this parallelism holds also for the results § 23.4.(4) and (5), the only

change being a constant multiplier $A(q_0)$ on the right-hand side of the inequalities; to inquire, that is, whether

(1) $$|q_n| < A(q_0),$$

and

(2) $$\Sigma |q_n|^2 \rho^{2n} < \frac{A(q_0)}{1-\rho}.$$

It will be found that we can prove (2). Our methods are essentially incapable of proving (1), and it seems likely to be false, but they do give

$$|q_n| < A(q_0) \log n \quad (n > 1).$$

There is a corresponding parallelism between functions R and functions $C(z)$, regular in γ, with missing values filling the half line $v = 0$, $u \leqslant 0$; and here we can actually prove our case.

A function C is of the form B^2. Hence if $\mathfrak{B}(\rho)$ and $\mathfrak{C}(\rho)$ are the majorants of $B(z)$ and $C(z)$, we have

$$\mathfrak{C}(\rho) \leqslant \mathfrak{B}^2(\rho),$$

the inequality, moreover, extending to the separate powers of ρ in the two expansions. We therefore deduce from the results for $\mathfrak{B}$,

(3) $$|C(z)| \leqslant \mathfrak{C}(\rho) \leqslant |c_0| \left(\frac{1+\rho}{1-\rho}\right)^2,$$

(4) $$|c_n| \leqslant 4|c_0| n \quad (n > 0).$$

[For (4) we gave a direct proof above in §23.6.] We have seen (Theorem 232) that (3) holds for functions R, save for a constant multiplier on the right, and we shall see presently that the same is true of (4).

26.4. We require the following results about $\bar{Q}(z) = \bar{Q}(z, q_0)$.

THEOREM 234.—

$$M_1(\rho, \mathfrak{R}\bar{Q}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\mathfrak{R}\bar{Q}(\rho e^{i\theta})| \, d\theta < A(q_0) \log \frac{2}{1-\rho}.$$

THEOREM 235.—

$$M_2^2(\rho, \bar{Q}) = \Sigma |\bar{q}_n|^2 \rho^{2n} < \frac{A(q_0)}{1-\rho}.$$

More generally $$M_r^r(\rho, \bar{Q}) < \frac{A(q_0, r)}{(1-\rho)^{r-1}} \quad (r > 1).$$

The proofs of these theorems also we postpone.

Since $|\mathfrak{R}Q|$ is subharmonic, it follows from Theorems 210 and 234 that

$$M_1(\rho, \mathfrak{R}Q) < A(q_0)\log\frac{2}{1-\rho}.$$

Hence, for $n > 0$,

$$|q_n\rho^n| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi} e^{-n\theta i}\,\mathfrak{R}Q\,d\theta\right| \leqslant 2M_1(\rho, \mathfrak{R}Q) < A(q_0)\log\frac{2}{1-\rho},$$

and by taking $\rho = 1-1/n$ in this we obtain†

THEOREM 236.— $$|q_n| < A(q_0)\log n \quad (n > 1).$$

From Theorems 210, Cor., and 235 we have, again,

THEOREM 237.—

$$M_2^2(\rho, Q) = \Sigma|q_n|^2\rho^{2n} < \frac{A(q_0)}{1-\rho}.$$

More generally $$M_r^r(\rho, Q) < \frac{A(q_0, r)}{(1-\rho)^{r-1}} \quad (r > 1).$$

This gives at once

THEOREM 238.— $$\mathfrak{Q}(\rho) < \frac{A(q_0)}{1-\rho} \quad (\rho < 1),$$

where $\mathfrak{Q}(\rho)$ is the majorant $\Sigma|q_n|\rho^n$ of $Q(z)$.‡

Thus the inequality for $|Q(z)|$ extends to $\mathfrak{Q}(\rho)$, in accordance with the behaviour of the functions B, and Theorem 238 is a generalization of Theorem 233.

Finally, since $R = Q^2$, we have (with $r_0 = q_0^2$)

$$M_1(\rho, R) = M_2^2(\rho, Q) \leqslant M_2^2(\rho, \bar{Q}) < \frac{A(r_0)}{1-\rho}.$$

From this there follows, by Theorem 213,

THEOREM 239.— $$|r_n| < A(r_0)\,n \quad (n > 0).$$

---

† For $M_1(\rho, Q)$ there is no better upper bound than $A(q_0)\log^2\frac{2}{1-\rho}$,

‡ In fact if $1-\rho_1 = \frac{1}{2}(1-\rho)$, we have

$$\Sigma|q_n|\rho^n \leqslant \{\Sigma|q_n|^2\rho_1^{2n}\,\Sigma(\rho/\rho_1)^{2n}\}^{\frac{1}{2}} < \left\{\frac{A(q_0)}{1-\rho_1}\,\frac{1}{\rho_1-\rho}\right\}^{\frac{1}{2}} < \frac{A(q_0)}{1-\rho}.$$

26.5. We take up now the postponed proofs of Theorems 233, 234, 235. These concern the modular function $\lambda(\tau)$; they involve rather heavy calculations, which the reader need not take too seriously if he is prepared to take results about classical functions for granted.

We need first some (very plausible) results about mean values taken on circles $|z| = \rho$ modified by a linear transformation

(1) $$\zeta = \frac{z+\alpha}{1+\alpha' z},$$

where $|\alpha| < 1$, and α' is the conjugate of α.

LEMMA. *Given an $f(z)$ regular in γ, let $F(z) = f(\zeta)$, where ζ is given by* (1). *Then*

(2) $$M_\lambda(\rho, F) \leqslant A(\alpha) M_\lambda(\rho^*, f) \quad (\lambda > 0),$$

(3) $$M(\rho, F) \leqslant M(\rho^*, f),$$

(4) $$M_k(\rho, \mathfrak{R}F) \leqslant A(\alpha) M_k(\rho^*, \mathfrak{R}f) \quad (k \geqslant 1),$$

where $\rho^* = 1-\beta(1-\rho)$, and β is a certain $A(\alpha)$.

Let $b = (1-|\alpha|)/(1+|\alpha|)$, and let us choose

$$\beta = \tfrac{1}{4}b, \quad \rho^* = 1-\beta(1-\rho).$$

Let $z = \rho e^{i\theta}$, $\zeta = re^{i\phi}$. The identity

$$\frac{1-|\zeta|^2}{1-|z|^2} = \frac{1-|\alpha|^2}{|1+\alpha' z|^2}.$$

shows that $1-r > \frac{1}{2}b(1-\rho)$, so that $r < \rho^*$ for all values of ρ, θ. Let now $Z = \rho^* e^{i\psi}$, and

$$P(Z, \zeta) = \frac{\rho^{*2}-r^2}{\rho^{*2}-2\rho^* r\cos(\psi-\phi)+r^2}.$$

When ρ and ψ are fixed, P is a function of θ (through ζ) satisfying

(5) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(Z, \zeta)\,d\theta = P\{Z, \zeta(0)\} = \frac{\rho^{*2}-|\alpha|^2}{|Z-\alpha|^2}\dagger$$
$$\leqslant \frac{\rho^*+|\alpha|}{\rho^*-|\alpha|} \leqslant \frac{2}{1-\frac{1}{4}b-|\alpha|} < A(\alpha).$$

† For $P\{Z, \zeta(z)\}$ is, for fixed Z, harmonic in z for $|z| \leqslant \rho$, and the left-hand side of (5) is therefore equal to the value of this at $z = 0$.

The function $|F(z)|^\lambda = |f(\zeta)|^\lambda$ is a subharmonic function of (Theorem 206); hence (Theorem 203)

$$|F(z)|^\lambda \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(Z)|^\lambda P(Z, \zeta)\, d\psi.$$

Integrating with respect to θ we have

$$\int_{-\pi}^{\pi} |F(z)|^\lambda\, d\theta \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(Z)|^\lambda \left(\int_{-\pi}^{\pi} P(Z, \zeta)\, d\theta\right) d\psi$$

$$\leqslant A(\alpha)\int_{-\pi}^{\pi} |f(Z)|^\lambda\, d\psi,$$

by (5). This proves (2). (4) can be proved in the same way, sinc $|\mathfrak{R} f(\zeta)|^k$ is subharmonic in ζ when $k \geqslant 1$. The result (3) is a immediate consequence of the inequality $r < \rho^*$.

26.6. Returning now to the function $\bar{Q}(z)$, we start from the formul

$$\kappa(\tau) = \log\lambda(\tau) = 4\log 2 + \pi\tau i + 8\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\frac{e^{m\pi\tau i}}{1+e^{m\pi\tau i}},$$

valid for a certain determination of the logarithm. Since $\bar{Q}(0) = q_0$ we have from the definition of $\bar{Q}$ (Theorem 231)

$$\bar{Q}(z) = q_0 - \kappa(\tau_0) + \kappa(\tau), \qquad \tau = \frac{\tau_0 - \tau_0' z}{1-z}.$$

Now
$$\phi(z) = e^{\pi\tau i} = \exp\left(\pi i\frac{\tau_0 - \tau_0' z}{1-z}\right)$$

satisfies $|\phi| \leqslant 1$ in γ, and $\phi(0) = a$, where a is a constant depending only on q_0. Hence $\phi(z)$ is subordinate to $\zeta(z)$, where

$$\zeta(z) = \frac{z+a}{1+a'z},$$

and
$$8\sum\frac{(-1)^m}{m}\frac{e^{m\pi\tau i}}{1+e^{m\pi\tau i}}$$

is subordinate to

$$\Psi(z) = \psi\{\zeta(z)\} = 8\sum\frac{(-1)^m}{m}\frac{\zeta^m(z)}{1+\zeta^m(z)}.$$

Thus
$$\bar{Q}(z) = L(z) + T(z),$$

where
$$L(z) = q_0 + \pi i(\tau - \tau_0) = q_0 - \frac{2\pi\mathfrak{I}(\tau_0)\, z}{1-z},$$

and $T(z)$ is subordinate to $\Psi(z) - \Psi(0)$.

26.7. The inequalities we have to prove are

(1) $$M(\rho, \bar{Q}) < \frac{A(q_0)}{1-\rho},$$

(2) $$M_r(\rho, \bar{Q}) < \frac{A(r, q_0)}{(1-\rho)^{1-1/r}} \quad (r > 1),$$

(3) $$M_1(\rho, \mathfrak{R}Q) < A(q_0) \log \frac{2}{1-\rho}.$$

Since $M_\lambda(g+h) \leqslant M_\lambda(g)+M_\lambda(h)$ when $\lambda \geqslant 1$, it is enough to prove the corresponding inequalities for $L(z)$ and $T(z)$. Those for $L(z)$ are easily verified.† To prove (1) and (2) for $T(z)$, it is enough, by Theorem 210, to prove them for $\Psi(z)$, and the same thing is true of (3), since $|\mathfrak{R}T|$ is subharmonic. The lemma shows finally, since $(1-\rho^*)^{-1} < A(a)(1-\rho)^{-1}$, that the inequalities are true for $\Psi(z)$ if they are true for $\psi(z)$. It remains, then, only to prove that

$$\psi(z) = 8\sum_1^\infty \frac{(-1)^m}{m} \frac{z^m}{1+z^m}$$

satisfies (1), (2), and (3) (with ψ in place of $\bar{Q}$).

26.8. Let $\rho = e^{-1/t}$, and let ν be the greatest integer contained in t. We have

(1) $$\psi = 8\left(\sum_1^\nu + \sum_{\nu+1}^\infty\right) \frac{(-1)^m}{m} \frac{z^m}{1+z^m} = 8(\psi_1+\psi_2),$$

say. For m of ψ_2, $\rho^m \leqslant e^{-1}$; hence

(2) $$|\psi_2| \leqslant A \sum_{\nu+1}^\infty \frac{1}{m} \rho^m < \frac{A}{\nu} \frac{1}{1-\rho} < A.$$

For m of ψ_1 we have $\rho^m \geqslant e^{-1}$, $1-\rho^m \geqslant (1-\rho)me^{-1}$. Hence

(3) $$|\psi_1| \leqslant A \sum_1^\nu \frac{1}{m(1-\rho^m)} \leqslant A(1-\rho)^{-1} \sum_1^\nu \frac{1}{m^2} = A(1-\rho)^{-1}.$$

---

† M is trivial. For M_r see § 8.4 (12) [$\alpha = 1$, $\beta = 0$], while § 8.4 (11) shows that

$$M_1(\rho, L) < A(q_0) \log\{2/(1-\rho)\}.$$

$M_1(\mathfrak{R}L)$ itself is bounded [by an $A(q_0)$], since $\mathfrak{R}L$ is substantially the kernel of a Poisson integral.

Also, for $r > 1$,

$$M_r(\rho, \psi_1) \leqslant A \sum_1^{\nu} M_r\left(\rho, \frac{1}{m}\frac{z^m}{1+z^m}\right) = A \sum_1^{\nu} \frac{1}{m}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{z^m}{1+z^m}\right|^r d\theta\right)^{1/r}$$

But if
$$z^m = -Z = -\rho^m e^{\Theta i},$$

$$\int_{-\pi}^{\pi}\left|\frac{z^m}{1+z^m}\right|^r d\theta = \int_{-\pi}^{\pi}\left|\frac{-Z}{1-Z}\right|^r d\Theta \leqslant \int_{-\pi}^{\pi} |1-Z|^{-r} d\Theta$$
$$< A(r)(1-\rho^m)^{-(r-1)}$$

by § 8.4 (12); and for m of ψ_1 this does not exceed $A(r)\{m(1-\rho)\}^{-(r-1)}$. Hence

(4) $$M_r(\rho, \psi_1) < A(r)(1-\rho)^{-(r-1)/r} \sum_1^{\nu} m^{-1-(r-1)/r} < A(r)(1-\rho)^{-1+1/r}.$$

Finally

$$M_1(\rho, \mathfrak{R}\psi_1) \leqslant \sum_1^{\nu} \frac{1}{m}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\mathfrak{R}\frac{z^m}{1+z^m}\right| d\theta,$$

$$\int_{-\pi}^{\pi}\left|\mathfrak{R}\frac{z^m}{1+z^m}\right| d\theta = \int_{-\pi}^{\pi}\left|\mathfrak{R}\frac{-Z}{1-Z}\right| d\Theta = \int_{-\pi}^{\pi}\left|\tfrac{1}{2}-\tfrac{1}{2}\frac{1-R^2}{1-2R\cos\Theta+R^2}\right| d\Theta$$
$$\leqslant \int_{-\pi}^{\pi}\left(\tfrac{1}{2}+\tfrac{1}{2}\frac{1-R^2}{1-2R\cos\Theta+R^2}\right) d\Theta = 2\pi,$$

and so

(5) $$M_1(\rho, \mathfrak{R}\psi_1) < A\sum_1^{\nu}\frac{1}{m} < A \log\frac{2}{1-\rho}.$$

The desired results for ψ follow from (1), (2), (3), (4), and (5).

26.9. A companion formula to that for $\kappa(\tau) = \log k^2(\tau) = \log\lambda(\tau)$ in § 26.6 is

$$\log k'^2(\tau) = -16 \sum_{m=0}^{\infty} \frac{1}{2m+1}\frac{q^{2m+1}}{1+q^{2(2m+1)}} \quad (q = e^{\pi i t})$$

valid for a certain determination of the logarithm. Replacing $e^{\pi\tau i}$ by z in this (as in § 25.2) we obtain the function [a companion of $\psi(z)$ of § 26.6]

$$\chi(z) = -16\sum_0^{\infty}\frac{1}{2m+1}\frac{z^{2m+1}}{1+z^{2(2m+1)}}.$$

The function $\chi(z)$ never takes any value $\pm 2n\pi i$ for $z \neq 0$ [compare $H(z)$ of § 25.2]; it differs from a Q by taking *one* of the forbidden values (viz. 0) just once. The reader can easily verify that the coefficient of

z^n in $\chi(z)$ is not $O(1)$; actually it can be as large as $A \log \log n$. Thus a result $q_n = O(1)$ is very improbable.

26.10. We conclude our study of functions Q by showing that the similar behaviour of the functions $\bar{B}$, $\bar{Q}$ breaks down in respect of means $M_\lambda(\rho)$ for which $0 < \lambda < 1$. In the first place we have (§ 8.4)

(1) $$M_\lambda(\rho, \bar{B}) \leqslant A(l, b_0) \quad (0 \leqslant \lambda \leqslant l < 1).$$

On the other hand

(2) $$M_\lambda(\rho, \bar{Q}) \neq O(1) \quad (\lambda > 0),$$

as $\rho \to 1$. A direct proof of (2) does not exist, and we argue indirectly as follows. If the mean is bounded (for any particular λ) it follows, by a known theorem, that $\bar{Q}$ tends almost always to a finite limit as z tends radially to the circumference of γ, further, that this limit cannot have the same value in any set of positive measure, and therefore differs almost always from all the values of any given denumerable set, in particular differs almost always from all numbers $\pm 2n\pi i$. Then $\bar{P} = e^{\bar{Q}}$ tends almost always to a finite limit other than 0 or 1. Now the interior of γ can be divided into fundamental regions for the function $\bar{P}$. These are curvilinear triangles with corners on $|z| = 1$, and the sides are of three types, corresponding to (real) values of w between $-\infty$ and 0, 0 and 1, 1 and $+\infty$ respectively. A radius vector to a point of $|z| = 1$ other than a corner crosses one, *and therefore two*, sides of each of an infinity of triangles. It crosses, therefore, an infinity of sides of at least two types; it follows that the limit of $\bar{P}$ can in general only be 0, 1, or ∞, and we arrive at a contradiction.

It is interesting to observe that we *can* prove

$$\int_0^{2\pi} |\mathfrak{R}\bar{Q}(\rho e^{i\theta})|^\lambda d\theta < A(l, q_0) \quad (0 \leqslant \lambda \leqslant l < 1),$$

so that the real and imaginary parts of $\bar{Q}$ behave differently. The proof, however, requires elaborations into which we cannot go.

27. *Functions S and $\bar{S}$.*

27.1. We define a function $\bar{S}$ to be any function $\bar{s}_1 z + \bar{s}_2 z^2 + \ldots$, regular and "schlicht" in γ, for which $\bar{s}_0 = \bar{S}(0) = 0$. We saw in § 11.2 that a "schlicht" $w = f(z)$ conformally represents γ on a simply-connected domain $\mathfrak{W}$ of the w-plane which does not overlap itself (is

"schlicht"). We define a function S to be any function that is subordinate to some $\bar{S}$.

We denote by Γ the set of "missing values" of $\bar{S}$ in γ.

By Theorem 217 (with $t=\rho=1$) the function $\bar{s}_1 z$ is either identical with $\bar{S}(z)$ or not subordinate to it, so that Γ contains at least one point w for which $|w| \leqslant |\bar{s}_1|$. Also it is the complement of a domain $\mathfrak{W}$ and is therefore closed. It is unbounded, and contains no bounded component Γ isolated from $\Gamma - \Gamma_1$, in particular no isolated point, since $\mathfrak{W}$ is simply connected. Thus Γ is perfect and connected. Since $w=0$ belongs to $\mathfrak{W}$, Γ has a positive distance d from 0. Also $d \leqslant |\bar{s}_1|$.

To sum up: Γ *is a connected (closed) continuum extending to infinity, and contains at least one point* w *for which* $0 < |w| = d \leqslant |\bar{s}_1|$.

In what follows Γ will always denote a connected continuum extending to ∞ and not containing $w=0$; $d(\Gamma)$ the distance of $w=0$ from Γ; $\Gamma(\bar{S})$ the set of missing values of $\bar{S}$.

THEOREM 240.—*The necessary and sufficient condition for a function, regular in* γ *and vanishing at* $z=0$, *to be an* S, *is that its set of missing values should contain a* Γ. *When* Γ *is given,* S *is subordinate to an* $\bar{S}$ *with* $\bar{s}_1 > 0$, *uniquely determined by* Γ, *whose set of missing values contains* Γ.

The condition is evidently necessary, since the superordinate $\bar{S}$ satisfies it. If, on the other hand, the condition is satisfied for a given Γ, the complementary set of Γ is a sum of *simply-connected* domains, each having as complete boundary part of Γ. One of these domains, $\mathfrak{W}$ say, contains $w=0$. Let $w=\Sigma(z)$ give the conformal representation of γ on $\mathfrak{W}$ with $\Sigma(0)=0$, $\Sigma'(0)>0$, so that Σ is uniquely determined. Γ is clearly a part of the set of missing values of Σ. Now any contour in γ beginning and ending with $z=0$ transforms by $w=S(z)$ into a contour lying in $\mathfrak{W}$, and beginning and ending at $w=0$ (otherwise the transform has a point in common with the boundary of $\mathfrak{W}$, and so with Γ). That is, S is subordinate to Σ. This completes the proof.

We may now abandon our original definition of functions S, and take instead: *A function* f *is said to be an* S *if* $f(0)=0$, *and the set of missing values of* f *in* γ *contains a (given) set* Γ.

We may for convenience write $\Gamma(S)$ for the Γ of the definition, and $\bar{S}(\Gamma)$ for the (unique) function $\bar{S}$ of Theorem 240.

We see then that we can obtain inequalities of types (A), (B), (C) for the general function S of the class just defined, provided we can obtain appropriate inequalities for the general function $\bar{S}$, that is, for the general function "schlicht" in γ (and vanishing at $z=0$). The subject of

functions $\bar{S}$ is of very great interest in itself, and we shall now study it systematically, not always confining ourselves to results that have a bearing upon the theory of functions S.

For convenience we shall state our results for functions $\bar{S}$ whose $\bar{s}_1$ is unity, and we shall write σ for such a function, σ_n for its n-th coefficient, so that $\sigma_0 = 0$, $\sigma_1 = 1$.

27.2. We begin with an important group of theorems†.

THEOREM 241.— $$|\sigma_2| \leqslant 2.$$

THEOREM 242.—

$$\frac{\rho}{(1+\rho)^2} \leqslant |\sigma(z)| \leqslant \frac{\rho}{(1-\rho)^2} \quad (|z| = \rho).$$

THEOREM 243.— $$d \geqslant \tfrac{1}{4}.$$

THEOREM 244.—

$$\frac{1-\rho}{(1+\rho)^3} \leqslant |\sigma'(z)| \leqslant \frac{1+\rho}{(1-\rho)^3}.$$

THEOREM 245.—

$$\frac{1-\rho}{1+\rho} \leqslant \left| z\,\frac{\sigma'(z)}{\sigma(z)} \right| \leqslant \frac{1+\rho}{1-\rho}.$$

The function

$$\sigma_0(z) = K(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \ldots$$

is a σ whose Γ is the set of points $u = 0$, $v \leqslant -\frac{1}{4}$ (§ 23.6). It shows that all the above inequalities are best possible. Theorem 241 turns out to be the key theorem, from which the whole group can be deduced fairly easily. We shall give two proofs for it; but it is convenient to postpone them, and to begin with the proof of the "deductions". The deduction of a Theorem Y from a Theorem X will be denoted by $(X \rightarrow Y)$.

27.21. *Proof of* (241 → 243). Let β be any point of $\Gamma(\sigma)$, Then

$$\frac{\sigma}{1-\sigma/\beta} = z + \left(\sigma_2 + \frac{1}{\beta}\right) z^2 + \ldots$$

is regular and "schlicht" in γ. Hence

$$|\sigma_2 + 1/\beta| \leqslant 2, \quad |1/\beta| \leqslant 2 + |\sigma_2| \leqslant 4, \quad |\beta| \geqslant \tfrac{1}{4};$$

and this is the desired result.

† They are fundamental in the general theory of the conformal representation of Riemann surfaces.

27.22. *Proof of* (241→244). This is an example of the "ζ-method" (§ 24.8).

Let $\phi(z) = \sigma(\zeta) - \sigma(z_0)$, where $\zeta = (z-z_0)/(z_0' z - 1)$. Then ϕ is an $\bar{S}$ with $s_1 = \sigma'(z_0)(d\zeta/dz)_0 = -(1-\rho^2)\sigma'(z_0)$, $\rho = |z_0|$, and

$$2\bar{s}_2 = (1-\rho^2)^2 \sigma''(z_0) - 2z_0^*(1-\rho^2)\sigma'(z_0).$$

The inequality $|\bar{s}_2| \leqslant 2|\bar{s}_1|$ becomes (if we drop the suffix 0)

(1) $$\left| (1-\rho^2) \frac{\sigma''(z)}{\sigma'(z)} - 2z' \right| \leqslant 4 \quad (|z| = \rho).$$

This is equivalent to

(2) $$\left| z \frac{\sigma''(z)}{\sigma'(z)} - \frac{2\rho^2}{1-\rho^2} \right| \leqslant \frac{4\rho}{1-\rho^2},$$

in which we have (for $z = \rho e^{i\theta}$)

$$\mathfrak{R}\, \frac{z\sigma''}{\sigma'} = \begin{cases} \rho(\partial/\partial\rho) \log|\sigma'| \\ (\partial/\partial\theta) \arg \sigma' \end{cases}, \quad \mathfrak{I}\, \frac{z\sigma''}{\sigma'} = \begin{cases} \rho(\partial/\partial\rho) \arg \sigma' \\ -(\partial/\partial\theta) \log|\sigma'| \end{cases},$$

and so, from (2),

(3) $$\frac{2\rho^2 - 4\rho}{1-\rho^2} \leqslant \rho \frac{\partial}{\partial\rho} \log|\sigma'| \leqslant \frac{2\rho^2+4\rho}{1-\rho^2},$$

(4) $$\frac{-4\rho}{1-\rho^2} \leqslant \rho \frac{\partial}{\partial\rho} \arg \sigma' \leqslant \frac{4\rho}{1-\rho^2}.$$

From (4) we can derive results concerning $\arg \sigma'$; we confine ourselves, however, to (3), which gives on division by ρ, integration and exponentiation [remember that $\sigma'(0) = 1$],

$$\frac{1-\rho}{(1+\rho)^3} \leqslant |\sigma'| \leqslant \frac{1+\rho}{(1-\rho)^3},$$

and this is Theorem 244.

27.23. *Proof of* (244→242). The right-hand inequality of 242 follows at once by integration from the right-hand inequality of 244. To deduce the left-hand inequality let $z = z_0$ be the point of $|z| = \rho$ for which $|\sigma|$ assumes its minimum value. This minimum increases with ρ [the image of $|z| \leqslant \rho$ by $w = \sigma(z)$ expands] and is less than d. Hence the *radius vector* from $w = 0$ to $w = \sigma(z_0)$ does not meet Γ, and there is a corresponding z-path from $z = 0$ to z_0. We take for parameter of this path

the distance r of the variable z from $z=0$. Since $|z_1-z_2| \geqslant |r_1-r_2|$, we have on the path (taking the limit) $|dz/dr| \geqslant 1$. Hence, if $|z_0| = \rho$,

$$|\sigma(z_0)| = (\textit{length of } w\textit{-path}) = \int_0^{z_0} |\sigma'(z)\, dz| = \int_0^{\rho} |\sigma'(z)| \left|\frac{dz}{dr}\right| dr.$$

But the integrand is at least $(1-r)\,(1+r)^{-3}.\,1$. Hence

$$|\sigma(z_0)| \geqslant \int_0^{\rho} \frac{1-r}{(1+r)^3}\, dr = \frac{\rho}{(1+\rho)^2}.$$

27.24. *Proof of* (242 → 245).—Consider again the function

$$\phi(z) = \sigma(\zeta) - \sigma(z_0).$$

where $\zeta = (z-z_0)/(z_0' z - 1)$ [§ 27.22]. Theorem 242 gives

$$(1-\rho^2)|\sigma'(z_0)| \frac{|z|}{(1+|z|)^2} \leqslant |\sigma(\zeta)-\sigma(z_0)| \leqslant (1-\rho^2)|\sigma'(z_0)| \frac{|z|}{(1-|z|)^2} \quad (\rho = |z_0|).$$

This becomes Theorem 245 when we put $\zeta = 0$ and so $z = z_0$.

27.31. We come now to the proof of the crucial Theorem 241. Our first proof depends on the following result, which is of interest in itself.

THEOREM 246.—*Let*

$$F(Z) = \sigma^{-1}(1/Z) = Z - \sigma_2 + a_1 Z^{-1} + a_2 Z^{-2} + \ldots,$$

so that $F(Z)$ is a function "schlicht" and regular (except at $Z=\infty$) in $|Z|>1$. Then

$$\sum_1^{\infty} n|a_n|^2 \leqslant 1.$$

As Z describes $|Z| = R > 1$ positively, σ describes negatively a closed (simple) contour, and F describes positively a closed (simple) contour C. Let $J = J(R)$ be the (positive) area of the interior of C, and let

$$F(Re^{i\theta}) = u(\theta) + iv(\theta).$$

By the formula for an area (note that the *sign* is correct: the point is vital to the argument)

$$J = \tfrac{1}{2}\int_0^{2\pi} (uv' - u'v)\, d\theta = \int_0^{2\pi} uv'\, d\theta$$

$$= \tfrac{1}{4}\int_0^{2\pi} \left\{(Re^{i\theta} + Re^{-i\theta}) - \mathfrak{R}\sigma_2 + \sum_1^{\infty}(a_n e^{-ni\theta} + \bar{a}_n e^{ni\theta}) R^{-n}\right\}$$

$$\times \left\{(Re^{i\theta} + Re^{-i\theta}) - \sum_1^{\infty} n(a_n e^{-ni\theta} + \bar{a}_n e^{ni\theta}) R^{-n}\right\} d\theta.$$

We need consider only terms of the product independent of θ, and obtain

$$J = \pi R^2 - \pi \sum_1^\infty n \,|\, a_n \,|^2 R^{-2n}.$$

Since on the one hand $J \geqslant 0$, and on the other R may be arbitrarily near 1, we must have

$$\sum_1^\infty n \,|\, a_n \,|^2 \leqslant 1.$$

Remarks. (1) As R increases C expands. For, by Theorem 117, Cor., the σ contour shrinks. (Or we may observe that two C's for different R's do not intersect, and C is approximately a large circle when R is large.) It can be shown that $\lim_{R\to 1} J(R)$ is the area of the region left uncovered in the w-plane when Z ranges over the whole exterior of the unit circle.

(2) A variant of the proof of Theorem 246 proceeds on lines which we may sketch as follows. We prove first that $J_1(R)$, the area of the image of $1 < |\, Z \,| < R$, is of the form

$$\pi R^2 - \pi + \pi \sum_1^\infty n \,|\, a_n \,|^2 + o(1),$$

when R is *large*. Next, it is not difficult to show that for large R $J(R)$ differs infinitesimally from the area of the ellipse $w = Z + a_1 Z^{-1}$, and so that $J(R) = \pi R^2 + o(1)$. Since $J \geqslant J_1$ we obtain the desired result when we make $R \to \infty$.

We can now deduce Theorem 241. Given a $\sigma(z)$, let

$$\phi(z) = \{\sigma(z^2)\}^{\frac{1}{2}} = z + \tfrac{1}{2}\sigma_2 z^3 + \ldots.$$

ϕ is regular and "schlicht" in γ. [$\phi(z) = a \neq 0$ has at most two solutions, of type $z = \pm\zeta$. But $\phi(-\zeta) = -\phi(\zeta)$, and $\pm\zeta$ cannot both give $\phi = a$.] Let $Z = z^{-1}$ and

$$F(Z) = 1/\phi(z) = Z - \tfrac{1}{2}\sigma_2 Z^{-1} + \ldots;$$

F is "schlicht" in $|\, Z \,| > 1$. By Theorem 246

$$|\, \tfrac{1}{2}\sigma_2 \,|^2 = |\, a_1 \,|^2 \leqslant \Sigma n \,|\, a_n \,|^2 \leqslant 1,$$

$$|\, \sigma_2 \,| \leqslant 2.$$

27.32. We proceed now to develop the ideas for another proof of Theorem 241.

THEOREM 247. *Suppose that $f(z)$ is regular in $|\, z \,| \leqslant r$. Let p be the maximum number, for varying w, of solutions in $|\, z \,| \leqslant r$ of $f(z) = w$.*

Let p_0 ($\leqslant p$) be the number of zeros of $f(z)$ in $|z| \leqslant r$. Let

$$M = \text{Max}\,|f|, \quad m = \text{Min}\,|f|$$

for $|z| = r$. Finally let $f(z) = Re^{i\Phi}$ and let $h(x)$ be a monotonic and absolutely continuous function of $x \geqslant 0$. Then (i), *if h is increasing we have*

(1) $$p_0 h(m) \leqslant I \leqslant p_0 h(m) + p\{h(M) - h(m)\};$$

(ii), *if h is decreasing we have*

(2) $$p_0 h(m) + p\{h(M) - h(m)\} \leqslant I \leqslant p_0 h(m),$$

where

$$I = \frac{1}{2\pi}\int_{|z|=r} h(R)\,d\Phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} h\{R(r, \theta)\}\,\frac{\partial\Phi(r, \theta)}{\partial\theta}\,d\theta.$$

Case (ii) reduces to case (i) if we change the sign of h. In case (i) we have $p_0 = \frac{1}{2\pi}\int_{|z|=r} d\Phi$ and so, defining

$$h_1(R) = \begin{cases} 0 & (R \leqslant m) \\ h(R) - h(m) & (R > m), \end{cases}$$

$$I_1 = I(h_1) = I - p_0 h(m).$$

Also $$I_1 = \frac{1}{2\pi}\int_{|z|=r} d\Phi \int_0^R \frac{h_1'(t)}{t}\,t\,dt.$$

This, on the one hand, is the integral of the non-negative function $h_1'(R)/R$ over the area (multiplicity counting) of the image of $|z| \leqslant r$ by $w = f(z)$, and is therefore non-negative; on the other, it does not exceed p times the same integral taken over the circle $|w| = M$. Thus

$$0 \leqslant I_1 \leqslant p\int_0^M h_1'(R)\,dR = p h_1(M),$$

which is equivalent to (1).

A more geometrical version of the proof may be sketched, taking for simplicity the case $p_0 = p = 1$. Consider the curve which is the image of $|z| = r$. It contains the w-origin O, and the radius vector from O meets the curve in an odd number of points $P_1, P_2, \ldots, P_{2n+1}$. As Φ increases points with odd suffixes move in the positive direction round the curve, points with even ones in the negative direction. The *sector* $d\Phi$ contributes to $2\pi I = \int h\,d\Phi$ an amount

$$2\pi\,dI = (h_1 - h_2 + \ldots - h_{2n} + h_{2n+1})\,d\Phi.$$

This lies between $h_1\,d\Phi$ and $h_{2n+1}d\Phi$, *a fortiori* between $h(m)d\Phi$ and $h(M)d\Phi$. Thus I lies between $h(m)$ and $h(M)$, the appropriate result.

COROLLARY.—*For a function $\sigma(z)$ we have*

$$\lambda m^{\lambda}(\rho) \leqslant \rho\,\frac{d}{d\rho}\,\{M_{\lambda}^{\lambda}(\rho)\} \leqslant \lambda M^{\lambda}(\rho) \qquad (\lambda > 0), \tag{3}$$

$$\rho\,\frac{d}{d\rho}\int_{-\pi}^{\pi}|\sigma(z)|^{-\beta}\,d\theta \leqslant 0 \qquad (\beta > 0). \tag{4}$$

For any $f(\rho e^{i\theta}) = Re^{i\Phi}$ we have, by the Cauchy-Riemann equations,

$$\frac{\partial\Phi}{\partial\theta} = \frac{\rho}{R}\,\frac{\partial R}{\partial\rho}.$$

Hence, by the main theorem with $p_0 = p = 1$, $h(x) = x^{\lambda}$ and $x^{-\beta}$ respectively, we have

$$\lambda m^{\lambda} \leqslant \rho\,\frac{d}{d\rho}\left(\frac{1}{2\pi}\int R^{\lambda}\,d\theta\right) = \frac{\lambda}{2\pi}\int R^{\lambda}\,d\Phi \leqslant \lambda M^{\lambda},$$

$$\rho\,\frac{d}{d\rho}\left(\int R^{-\beta}\,d\theta\right) = -\beta\int R^{-\beta}\,d\Phi \leqslant 0,$$

which are the desired results.

To prove Theorem 241 let now

$$f(z) = \left(\frac{\sigma(z)}{z}\right)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}\sigma_2 z + \ldots,$$

a function regular in γ. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\sigma|^{-1}\,d\theta = \rho^{-1}\,\frac{1}{2\pi}\int|f|^{2}\,d\theta = \rho^{-1}\left(1 + \tfrac{1}{4}|\sigma_2|^2\rho^2 + \sum_{n=2}^{\infty}k_n\rho^{2n}\right),$$

where $k_n \geqslant 0$. The differential coefficient of this with respect to ρ is non-positive, by (2) of the Corollary, with $\beta = 1$. Hence

$$-\rho^{-2} + \tfrac{1}{4}|\sigma_2|^2 + \sum_{2}^{\infty}(2n-1)k_n\rho^{2n-2} \leqslant 0,$$

$$|\sigma_2|^2 \leqslant 4\rho^{-2},$$

and so $|\sigma_2|^2 \leqslant 4$.

27.41. So far we have succeeded in proving best possible results in all cases. Our remaining theorems about functions σ are (in general) not best possible.

THEOREM 248.—

$$M_1(\rho, \sigma) \leqslant \frac{\rho}{1-\rho},$$

$$|\sigma_n| < en.$$

Proof (i). Clearly

$$\phi(z) = \{\sigma(z^2)\}^{\frac{1}{2}} = z + b_2 z^2 + b_3 z^3 + \ldots$$

is regular and "schlicht" in γ, and in virtue of Theorem 242 (applied to σ),

(1) $$|\phi(\rho e^{i\theta})| \leqslant \frac{\rho}{1-\rho^2}.$$

Now

$$\pi \sum_1^\infty n|b_n|^2 \rho^{2n} = \int_0^\rho r\,dr \left(2\pi \sum_1^\infty |nb_n|^2 r^{2n-2}\right) = \int_0^\rho r\,dr \int_0^{2\pi} |\phi'(re^{i\theta})|^2 d\theta$$

$$= \{\textit{area of the transform of } |z| \leqslant \rho \textit{ by } w = \phi(z)\}$$

(2) $$\leqslant \pi M^2(\rho, \phi),$$

since ϕ is 'schlicht", so that the area does not exceed π times the square of the greatest radius from $w = 0$. Hence

$$\sum_1^\infty n|b_n|^2 \rho^{2n-1} \leqslant \frac{\rho}{(1-\rho^2)^2}.$$

Integrating this from 0 to ρ we obtain

(3) $$\sum_1^\infty |b_n|^2 \rho^{2n} \leqslant \frac{\rho^2}{1-\rho^2}.$$

But

$$\int_0^{2\pi} |\sigma(\rho^2 e^{i\psi})|\,d\psi = \int_0^{2\pi} |\sigma(\rho^2 e^{2i\theta})|\,d\theta$$

$$= \int_0^{2\pi} |\phi(\rho e^{i\theta})|^2 d\theta = 2\pi \sum_1^\infty |b_n|^2 \rho^{2n}.$$

Hence (3) becomes

$$2\pi M_1(\rho^2, \sigma) \leqslant \frac{2\pi\rho^2}{1-\rho^2}.$$

We may write ρ for ρ^2 in this, obtaining the first part of the theorem.

The second part follows by the usual inference from M_1 to c_n.

27.43. *Proof* (ii). This extends naturally to prove the more general result (obtainable also by the other method):

THEOREM 249.—

(1) $$\lambda \int_0^\rho m^\lambda(r) \frac{dr}{r} \leqslant M_\lambda^\lambda(\rho, \sigma) \leqslant \lambda \int_0^\rho M^\lambda(r) \frac{dr}{r} \quad (\lambda > 0).$$

(2) $$2\lambda\pi \int_0^\rho r^{\lambda-1}(1+r)^{-2\lambda} dr \leqslant M_\lambda^\lambda(\rho, \sigma) \leqslant 2\lambda\pi \int_0^\rho r^{\lambda-1}(1-r)^{-2\lambda} dr.$$

In particular

(3) $$M_\lambda(\rho, \sigma) < A(\lambda)(1-\rho)^{-2\lambda+1} \quad (\lambda > \tfrac{1}{2}),$$

(4) $$M_\lambda(\rho, \sigma) < A(\lambda) \quad (\lambda < \tfrac{1}{2}).$$

(1) follows by integration from Theorem 247, Cor. (3), and (2) from (1) and Theorem 242. It is interesting to observe in connexion with the second inequality in (1) that for *any* function regular in γ and vanishing at the origin we have the opposite inequality

$$\left(\frac{M_\lambda(\rho)}{\rho}\right)^\lambda \geqslant \frac{1}{\pi} \int_0^\rho \left(\frac{M(r)}{r}\right)^\lambda dr.$$

It follows from (4), the theorem mentioned in §26.10, and Theorem 210, Cor., that functions S and $\bar{S}$ tend almost always to a limit as z tends radially to the circumference of γ.

We actually use† the result for $\bar{S}$ in a moment, and therefore digress to give an *ad hoc* proof of it.

If β is a missing value of $\bar{S}$ and $\phi = (\bar{S}-\beta)^{\frac{1}{4}}$ (say), it follows from (4) that $M_2(\phi) = O(1)$. It is therefore enough to prove that a function u, harmonic in γ and satisfying $M_2(u) = O(1)$, has a p.p. radial limit. If

$$u = \Sigma(a_n \cos n\theta + b_n \sin n\theta)\rho^n,$$

the condition $M_2(u) = O(1)$ is $\Sigma(|a_n|^2+|b_n|^2)\rho^{2n} = O(1)$, equivalent to $\Sigma(|a_n|^2+|b_n|^2)$ convergent. By the Riesz-Fischer Theorem (Theorem 44) $\Sigma(a_n \cos n\theta + b_n \sin n\theta)$ is the Fourier series of a function U (of L^2), and by Theorem 36 $\sigma_n(\theta) \to U(\theta)$ p.p. This is a stronger result than the p.p. convergence of u to U as $\rho \to 1$. (The "Abel limit" exists if the Cesaro mean converges, the proof being a summation twice by parts.)

† Unexpectedly!

27.43.—THEOREM 250.—*The perimeter of the transform of* $|z| = \rho$ *by* $w = \sigma(z)$ *does not exceed* $\dfrac{A\rho}{(1-\rho)^2}$.

The perimeter being $\int_0^{2\pi} |\sigma'(\rho e^{i\theta})| \rho \, d\theta = \rho M_1(\rho, \sigma')$, our result follows from Theorem 248 and the following lemma:—

If $f(z)$ *is regular in* γ, *and*

$$M_1(\rho, f) \leqslant K(1-\rho)^{-\alpha} \quad (\alpha \geqslant 0),$$

then $$M_1(\rho, f') \leqslant A(\alpha) K (1-\rho)^{-\alpha-1}.\dagger$$

Let $1-\rho_1 = \frac{1}{2}(1-\rho)$, $x = \rho e^{i\theta}$, $z = \rho_1 e^{i\phi}$. We have

$$\int_0^{2\pi} |f'(x)| \, d\theta = \int_0^{2\pi} d\theta \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z)}{(z-x)^2} iz d\phi \right| \leqslant A \int_0^{2\pi} d\phi \, |f(z)| \int_0^{2\pi} \frac{d\theta}{|z-x|^2}$$

$$\leqslant A \int_0^{2\pi} d\phi \, |f(z)| \frac{A}{\rho_1 - \rho} \leqslant \frac{A}{1-\rho} M_1(\rho_1, f),$$

and $$M_1(\rho_1, f) \leqslant K(1-\rho_1)^{-\alpha} \leqslant KA(\alpha)(1-\rho)^{-\alpha}.$$

Alternatively we can argue from Theorems 245 and 248 (without the lemma):

$$M_1(\rho, \sigma') = \frac{1}{2\pi} \int \left| \frac{\sigma'}{\sigma} \cdot \sigma \right| d\theta \leqslant \frac{\rho^{-1}(1+\rho)}{1-\rho} M_1(\rho, \sigma) \leqslant \frac{1+\rho}{(1-\rho)^2}.$$

27.44. Let $\mu(z) = z + \mu_3 z^3 + \mu_5 z^5 + \ldots$ be an odd function, "schlicht" in γ. $\mu^2(z) = z^2 + 2\mu_3 z^4 + \ldots$ has different values for different values of z^2 in γ. Hence $\mu(z) = \{\sigma(z^2)\}^{\frac{1}{2}}$, where

$$\sigma(z) = z + 2\mu_3 z^2 + \ldots$$

is "schlicht" in γ. Conversely, for any σ, the function μ so defined is an odd schlicht function. This relation enables us to deduce properties of functions μ from those of general σ. For example, Theorem 241 gives $|\mu_3| \leqslant 1$: this is a best possible result, as is shown by $\mu(z) = z/(1-z^2)$. We prove now

THEOREM 251.— $$|\mu_n| \leqslant A_1.$$

The best possible A_1 is not known. It is known, however, that for any assigned odd $n > 3$ it is possible to have $|\mu_n| > 1$. To prove

† If $\alpha = 0$ it is possible to prove a little more, viz. that

$$M_1(\rho, f') = o\{(1-\rho)^{-1}\}.$$

Theorem 251 consider $\mu_1(z)=\{\mu(z^2)\}^{\frac{1}{2}}$ and $\mu_2(z)=\{\mu_1(z^2)\}^{\frac{1}{2}}$. $\mu_1(z)$ and so $\mu_2(z)$ are "schlicht" in γ. Also

$$\mu_2(z)=\{\mu(z^4)\}^{\frac{1}{4}}=\{\sigma(z^8)\}^{\frac{1}{8}},$$

where $\sigma(z)$ is "schlicht". If $z=\rho e^{i\theta}$, and $\rho \geqslant \frac{1}{2}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\mu'(z)|\,d\theta=\frac{1}{8\pi}\int_{-4\pi}^{4\pi}=\frac{\rho^{-\frac{3}{4}}}{8\pi}\int_{-4\pi}^{4\pi}|\mu_2(z^{\frac{1}{4}})^3\mu_2'(z^{\frac{1}{4}})|\,d\theta$$

$$\leqslant \rho^{-\frac{3}{4}}\left(\frac{1}{8\pi}\int_{-4\pi}^{4\pi}|\mu_2(z^{\frac{1}{4}})|^6\,d\theta\right)^{\frac{1}{2}}\left(\frac{1}{8\pi}\int_{-4\pi}^{4\pi}|\mu_2'(z^{\frac{1}{4}})|^2\,d\theta\right)^{\frac{1}{2}}$$

$$=\rho^{-\frac{3}{4}}P^{\frac{1}{2}}Q^{\frac{1}{2}}.$$

By Theorem 249 (3),

$$P=\frac{1}{8\pi}\int_{-4\pi}^{4\pi}|\sigma(z^2)|^{\frac{3}{4}}\,d\theta=\frac{1}{16\pi}\int_{-8\pi}^{8\pi}|\sigma(\rho^2e^{it})|^{\frac{3}{4}}\,dt=\frac{1}{2\pi}\int_{-\pi}^{\pi}$$

(1) $$<A(1-\rho^2)^{-\frac{3}{2}+1}<A(1-\rho)^{-\frac{1}{2}}.$$

Now $$Q=\frac{1}{2\pi}\int_{-\pi}^{\pi}|\mu_2{}'(\rho^{\frac{1}{4}}e^{it})|^2\,dt$$

increases with ρ, so that

$$Q\leqslant\frac{1}{\pi(\rho^{\frac{1}{4}}-\rho^{\frac{1}{8}})}\int_{\rho^{\frac{1}{4}}}^{\rho^{\frac{1}{8}}}r\,dr\int_{-\pi}^{\pi}|\mu_2'(re^{it})|^2\,dt.$$

Hence, by (2) in § 27.41† and Theorem 242,

$$Q<A(1-\rho)^{-1}M^2(\rho^{\frac{1}{8}},\mu_2)<A(1-\rho)^{-1}M^{\frac{1}{4}}(\rho,\sigma)$$

(2) $$<A(1-\rho)^{-1}(1-\rho)^{-\frac{1}{2}}<A(1-\rho)^{-\frac{3}{2}}.$$

It follows from (1) and (2) that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\mu'(z)|\,d\theta<A(1-\rho)^{-1},$$

and so, taking $\rho=1-1/n$, $n>1$,

$$n|\mu_n|\leqslant\rho^{-n+1}\frac{1}{2\pi}\int_{-\pi}^{\pi}|\mu'(z)|\,d\theta\leqslant A(1-\rho)^{-1}=An,$$

the desired result.

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† Valid for any "schlicht" ϕ. We take $\phi=\mu_2$, and $\rho^{\frac{1}{4}}$ for ρ.

27.5. We have seen (Theorem 241) that $|\sigma_2| \leqslant 2$. It can also be proved (by a long and difficult argument) that $|\sigma_3| \leqslant 3$. Since $|\sigma_n| < en$, by Theorem 248, it is natural to inquire whether $|\sigma_n| \leqslant n$ is true generally. (No stronger result is possible, as the example $\sigma_0(z)$ shows.) This problem is still unsolved†. It is therefore interesting that we can answer it in the affirmative for two important sub-classes of functions σ (the function $\sigma_0(z)$ belonging to both of them).

27.51. THEOREM 251.—*Suppose that $\sigma(z)$ has real coefficients. Then*

Since σ is real on the real axis of z, the image D of γ by $w = \sigma(z)$ is divided symmetrically by the real axis of w, and the two halves are images of the two z semi-circles. Thus

$$v(\rho, \theta) = \mathfrak{I}\sigma(\rho e^{i\theta}) = \sum_1^\infty \sigma_n \rho^n \sin n\theta \geqslant 0 \quad (0 \leqslant \theta \leqslant \pi),$$

(this being the sense of the inequality for small ρ). Now $|\sin n\theta / \sin\theta| \leqslant n$, and so

$$(1) \quad \rho^n |\sigma_n| = \frac{2}{\pi} \left| \int_0^\pi v(\rho, \theta) \sin n\theta \, d\theta \right| \leqslant \frac{2n}{\pi} \int_0^\pi v(\rho, \theta) \sin\theta \, d\theta = n\sigma_1 = n.$$

This, in the limit $\rho \to 1$, gives the desired result.

27.52. A domain $\mathfrak{W}$ (containing $w = 0$) is called "starshaped" (with respect to $w = 0$) if, for any point w of $\mathfrak{W}$, the whole radius vector from 0 to w belongs to $\mathfrak{W}$. A closed (simple) contour is called "starshaped" if its interior is a "starshaped" domain.

The following results are so striking and complete that we include them, but the proofs are rather difficult, and the reader may omit them if he wishes.

THEOREM 252.—*Suppose that $\sigma(z)$ transforms γ into a "starshaped" domain. Then $|\sigma_n| \leqslant n$.*

The proof of this is based on

THEOREM 253.—*For $\sigma(z)$ to transforms γ into a "starshaped" domain it is necessary and sufficient that, in γ,*

$$\mathfrak{R} z \frac{\sigma'(z)}{\sigma(z)} > 0.$$

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† The corresponding result for function $\mu(z)$, viz. $|\mu_n| \leqslant 1$, is false. There are, however, differences in the two cases which make this no presumption against the truth of $|\sigma_n| \leqslant n$.

COROLLARY.—*If $\mathfrak{W}(\sigma)$ is "starshaped" the transform by $\sigma(z)$ of each circle $|z|=\rho$ is "starshaped".*

We must use the result noted in § 27.42, that $\sigma(\rho e^{i\theta})\to\sigma(\theta)$ as $\rho\to 1$ for almost all θ. $\sigma(\theta)$ is a point of Γ (see § 27.1), and $\sigma(\theta)\neq 0$.

Let $0\leqslant\theta<2\pi$, and define $\Phi(\rho,\theta)=\arg\sigma(\rho e^{i\theta})$ by continuation along the radius $\rho e^{i\theta}$ starting with $\Phi(0,\theta)=\theta$. Then $\Phi(\theta)=\lim\Phi(\rho,\theta)$ exists and is finite for a p.p. set E of θ. For a θ of E we denote by $c_1(\theta)$ the transform of the radius $z=\rho e^{i\theta}$, $0\leqslant\rho\leqslant 1$. $c_1(\theta)$ joins $w=0$ to $\sigma(\theta)$. We denote by $c_2(\theta)$ the "ray" $w=\lambda\sigma(\theta)$, $\lambda\geqslant 1$. If $\mathfrak{W}(\sigma)$ is "starshaped" $c_1(\theta)$ and $c_2(\theta)$ have only the point $\sigma(\theta)$ in common, so that $c(\theta)=c_1(\theta)+c_2(\theta)$ is a simple curve extending from $w=0$ to infinity. Let θ_1, θ_2 be θ of E satisfying $0\leqslant\theta_1<\theta_2<2\pi$. The curve $c_1(\theta_2)$ has with $c(\theta_1)$ (besides $w=0$) at most the point $\sigma(\theta_2)$ in common, which then must be on $c_2(\theta_1)$. From this [and our definition of $\Phi(\rho,\theta)$] we conclude that for θ_1, θ_2 of E

$$(1)\qquad \min_{\rho\leqslant 1}\Phi(\rho,\theta_1)<\Phi(\rho,\theta_2)<\max_{\rho\leqslant 1}\Phi(\rho,\theta_1)+2\pi,$$

the extremes being finite, and also that

$$(2)\qquad \Phi(\theta_1)\leqslant\Phi(\theta_2)\leqslant\Phi(\theta_1)+2\pi.$$

From (1) we deduce further that $|\Phi(\rho,\theta)|\leqslant M$ for all $\rho<1$ and all θ†.

We may assume, without loss of generality, that $\Phi(0)$ exists. Now

$$(3)\qquad \Re z\frac{\sigma'(z)}{\sigma(z)}=\frac{\partial}{\partial\theta}\Phi(\rho,\theta).$$

Hence, if $z=\rho e^{i\theta}$ and $\rho<r<1$, Poisson's formula gives

$$\Re z\frac{\sigma'(z)}{\sigma(z)}=\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial\psi}\Phi(r,\psi)\frac{r^2-\rho^2}{r^2-2r\rho\cos(\psi-\theta)+\rho^2}d\psi$$

$$=\frac{r^2-\rho^2}{r^2-2r\rho\cos\theta+\rho^2}-\frac{1}{2\pi}\int_0^{2\pi}\Phi(r,\psi)\frac{\partial}{\partial\psi}\frac{r^2-\rho^2}{r^2-2r\rho\cos(\psi-\theta)+\rho^2}d\psi.$$

Since $\Phi(r,\psi)$ is uniformly bounded, we may (making $r\to 1$) replace r by 1, obtaining

$$\Re z\frac{\sigma'(z)}{\sigma(z)}=\frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2}-\frac{1}{2\pi}\int_0^{2\pi}\Phi(\psi)\frac{\partial}{\partial\psi}\frac{1-\rho^2}{1-2\rho\cos(\psi-\theta)+\rho^2}d\psi.$$

† In the first instance for all θ of E, but the restriction to E (for a fixed ρ) can be dropped!

Now by (2) $\Phi(\psi)$ increases from $\Phi(0)$ to $\Phi(0)+2\pi$ as ψ increases (in E) from 0 to 2π. Hence, for a certain mean value ξ,

$$\mathbf{R} z \frac{\sigma'(z)}{\sigma(z)} = \frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2}$$
$$-\frac{1}{2\pi}\left[\Phi(0)\int_0^{\xi}+\left(\Phi(0)+2\pi\right)\int_{\xi}^{2\pi}\frac{\partial}{\partial\psi}\frac{1-\rho^2}{1-2\rho\cos(\psi-\theta)+\rho^2}\,d\psi\right]$$
$$=\frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2}-\int_{\xi}^{2\pi}=\frac{1-\rho^2}{1-2\rho\cos(\xi-\theta)+\rho^2}>0.$$

This proves the necessity part of our Theorem.

To prove the sufficiency, we note that the inequality $\mathbf{R} z\sigma'(z)/\sigma(z)>0$ implies†, by (3), that $\Phi(\rho,\theta)$, for fixed ρ, increases as θ increases. Since the transform $c(\rho)$ of $|z|=\rho$ by $\sigma(z)$ is a simple contour including $w=0$, this again evidently implies† that $c(\rho)$ is "starshaped". This, combined with the necessity part, proves the corollary. Finally, since any w belonging to $\mathfrak{W}(\sigma)$ is inside $c(\rho)$ for sufficiently large ρ, the whole radius from 0 to w will then be inside $c(\rho)$ and so in $\mathfrak{W}(\sigma)$, *i.e.* $\mathfrak{W}(\sigma)$ is "starshaped".

It is now easy to prove Theorem 252. Since $\sigma_1=1$, we can proceed by induction. Let $n>1$ and let us assume that $|\sigma_k|\leqslant k$ for all $k\leqslant n-1$. Let

$$f(z)=z\frac{\sigma'(z)}{\sigma(z)}=1+c_1z+c_2z^2+\dots.$$

From $\sigma f=z\sigma'$ we obtain, equating coefficients of z^n,

$$(n-1)\sigma_n=c_1\sigma_{n-1}+c_2\sigma_{n-2}+\dots+c_{n-1}.$$

By Theorem 253 $\mathbf{R} f(z)>0$ in γ. This implies $|c_n|\leqslant 2$ [Theorem 110]. Hence

$$(n-1)|\sigma_n|\leqslant 2[(n-1)+(n+2)+\dots+1]=n(n-1),$$

the desired result.

27.53. A domain $\mathfrak{W}$ is called convex if, for any two points w_1 and w_2 of $\mathfrak{W}$, the whole stretch joining w_1 and w_2 belongs to $\mathfrak{W}$. It follows that, for any n points $w_1, w_2, \dots, w_n$ of a convex $\mathfrak{W}$, their arithmetic mean

$$w=(w_1+w_2+\dots+w_n)/n$$

belongs to $\mathfrak{W}$.

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† Actually "is equivalent to" (in both cases marked †).

THEOREM 254.—*Suppose that $\sigma(z)$ transform γ into a convex domain. Then $|\sigma_n| \leqslant 1$.*

This is a best possible result, since $z(1-z)^{-1} = z+z^2+z^3+\ldots$ transforms γ into the (convex) half-plane $u > -\frac{1}{2}$.

The proof of Theorem 254 is much the same as proof (iv) of Theorem 110. If $\omega = e^{2\pi i/n}$,

$$\sigma_n z+\sigma_{2n} z^2+\ldots = \frac{1}{n}\sum_{r=0}^{n-1} \sigma(\omega^r z^{1/n})$$

is subordinate to $\sigma(z)$, since the right-hand side is the arithmetic mean of n points of $\mathfrak{W}$. Hence $|\sigma_n| \leqslant |\sigma_1| = 1$, by Theorem 212.

27.61. **From the theorems about functions σ we can draw conclusions about functions S.**

THEOREM 255.—

$$M(\rho, S) \leqslant \frac{\bar{s}_1\rho}{(1-\rho)^2} \leqslant \frac{4d\rho}{(1-\rho)^2},$$

where $\bar{s}_1 = \bar{s}_1(\Gamma)$ and $d = d(\Gamma)$, as defined in § 25 . 1.

By Theorem 240 S is subordinate to an $\bar{S}$ with $\bar{s}_1 > 0$ whose $\Gamma(\bar{S})$ contains Γ. Hence $d(\Gamma) \geqslant d\{\Gamma(\bar{S})\} = d(\bar{S}) \geqslant \frac{1}{4}\bar{s}_1$, by Theorem 243. Theorem 255 now follows from Theorem 242.

From Theorem 248 we deduce

THEOREM 256.—

$$M_1(\rho, S) \leqslant \frac{\bar{s}_1\rho}{1-\rho} \leqslant \frac{4d\rho}{1-\rho},$$

$$|s_n| < e\bar{s}_1 n \leqslant 4edn,$$

where $\bar{s}_1 = \bar{s}_1(\Gamma)$ and $d = d(\Gamma)$.

For $\bar{s}_1 \leqslant 4d$, and we use Theorems 213 and 248.

THEOREM 257.— $$|s_2| \leqslant 2\bar{s}_1 \leqslant 8d.$$

For, by Theorems 212 and 241,

$$|s_2| \leqslant \text{Max}\,(\bar{s}_1, |\bar{s}_2|) \leqslant 2\bar{s}_1 \leqslant 8d.$$

27.62. It is another open question whether $|s_n| \leqslant n|\bar{s}_1|$ whenever S is subordinate to $\bar{S}$; the function $\sigma_0(z)$ (subordinate to itself) shows that more cannot be true We actually know it to be true for $n = 1$ and

$n = 2$ (Theorems 212 and 257). For general s_n we shall now prove some results similar to those of § 27.5 for σ_n. We begin with

THEOREM 258.—*Suppose that $\bar{S}(z)$ transform γ into a convex domain. If $S(z)$ is subordinate to $\bar{S}(z)$, then $|s_n| \leqslant |\bar{s}_1|$.*

The proof is exactly the same as that of Theorem 254.

27.63. We need the following results, which are of considerable intrinsic interest and have other applications†.

THEOREM 259.—*Let $|S_n(\theta)| \leqslant 1$ for all real θ, where*

$$S_n(\theta) = a_1 \sin\theta + a_2 \sin 2\theta + \ldots + a_n \sin n\theta.$$

Then
$$\left|\frac{S_n(\theta)}{\sin\theta}\right| \leqslant n.$$

COROLLARY 1.—

$$|a_1 + 2a_2 + \ldots + na_n| \leqslant n.$$

COROLLARY 2.—*Let $|Q_n(z)| \leqslant 1$ for all z in γ, where*

$$Q_n(z) = b_0 + b_1 z + \ldots + b_n z^n.$$

Then $|Q_n'(z)| \leqslant n$ in γ.

Both the theorem and the corollaries are "best possible", as the examples $S_n(\theta) = \sin n\theta$ and $Q_n(z) = z^n$ show.

We may assume $0 \leqslant \theta \leqslant \pi$. If $\frac{1}{2}\pi/n \leqslant \theta \leqslant \frac{1}{2}\pi$, then $\sin\theta \geqslant 2\theta/\pi \geqslant 1/n$, and so $|(\sin\theta)^{-1} S_n(\theta)| \leqslant n$. Similarly if $\frac{1}{2}\pi \leqslant \theta \leqslant \pi - \frac{1}{2}\pi/n$. Let now $0 \leqslant \theta \leqslant \frac{1}{2}\pi/n$ or $\pi - \frac{1}{2}\pi/n \leqslant \theta \leqslant \pi$. Let $\cos\theta = x$. The function $\cos n\theta$ is a polynomial $T_n(x)$ of degree n, whose zeros are at $x_k = \cos\theta_k$, where $\theta_k = \frac{1}{2}(2k-1)\pi/n$, $1 \leqslant k \leqslant n$. Also

$$T_n'(x) = n\frac{\sin n\theta}{\sin\theta}; \qquad |T_n'(x)| \leqslant n^2. \tag{1}$$

This formula shows that $(\sin\theta)^{-1} S_n(\theta)$ is a polynomial in x of (at most) degree $n-1$. Hence, using Lagrange's interpolation formula,

$$\frac{S_n(\theta)}{\sin\theta} = \sum_{k=1}^{n} \frac{S_n(\theta_k)}{\sin\theta_k} \frac{1}{T_n'(x_k)} \frac{T_n(x)}{x - x_k} = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k-1} S_n(\theta_k) \frac{T_n(x)}{x - x_k}.$$

† Corresponding inequalities with an extra factor A on the right-hand side are comparatively trivial, the essential exact ones not at all.

Now for our θ we have $x_1 \leqslant x \leqslant 1$ or $-1 \leqslant x \leqslant x_n$, respectively, so that $T_n(x)/(x-x_k)$ is of constant sign for all k. Hence

$$\left|\frac{S_n(\theta)}{\sin\theta}\right| \leqslant \frac{1}{n}\left|\sum_{k=1}^{n} \frac{T_n(x)}{x-x_k}\right| = \frac{1}{n}\,|T'_n(x)| \leqslant n.$$

This completes the proof.

Corollary 1 is the limiting case $\theta = 0$. As for Corollary 2,

$$S_n(\theta) = \frac{Q_n(\zeta e^{i\theta}) - Q_n(\zeta e^{-i\theta})}{2i} = b_1\zeta\sin\theta + \ldots + b_n\zeta^n \sin n\theta \qquad (|\zeta| = 1)$$

satisfies $|S_n(\theta)| \leqslant 1$, and so, by Corollary 1,

$$|Q'(\zeta)| = |\zeta Q'(\zeta)| = |b_1\zeta + 2b_2\zeta^2 + \ldots + nb_n\zeta^n| \leqslant n.$$

This (and the maximum modulus principle) proves Corollary 2.

27.64. THEOREM 260.—*Suppose that $\bar{S}(z)$ has real coefficients $\bar{s}_n$. If $S(z)$ is subordinate to $\bar{S}(z)$ then* $|s_n| \leqslant n|\bar{s}_1|$.

We recall [§ 22.5, Lemma 4 and (3)] that, if $S(z) = \bar{S}\big(\omega(z)\big)$,

$$s_n = \sum_{k=1}^{n} a_n^{(k)}\bar{s}_k; \quad \omega^k(z) = \sum_{n=k}^{\infty} a_n^{(k)} z^n;$$

and that in γ

$$|P_n(z)| = \left|\sum_{k=1}^{n} a_n^{(k)} z^k\right| \leqslant 1, \qquad \left|z^n P_n\left(\frac{1}{z}\right)\right| \leqslant 1.$$

We may assume that $\bar{s}_1 > 0$. Then by § 27.51 (1),

$$s_n = \frac{2}{\pi}\int_0^{\pi} v(\rho,\theta) \sum_{k=1}^{n} a_n^{(k)}\rho^{-k}\sin k\theta\, d\theta = \frac{2}{\pi}\int_0^{\pi} v(\rho,\theta)\, S_n(\theta)\, d\theta,$$

where $v(\rho,\theta) = \mathfrak{I}\bar{S}(z)$,

$$|S_n(\theta)| = \left|\sum_{k=1}^{n} a_n^{(k)}\rho^{-k}\sin k\theta\right| = \left|\frac{P_n(1/z) - P_n(1/z')}{2i}\right| \leqslant \rho^{-n} \qquad (z = \rho e^{i\theta})$$

and $v(\rho,\theta) \geqslant 0$ if $0 \leqslant \theta \leqslant \pi$. Hence, using Theorem 259,

$$|s_n| \leqslant \frac{2n\rho^{-n}}{\pi}\int_0^{\pi} v(\rho,\theta)\sin\theta\, d\theta = n\rho^{-(n-1)}\bar{s}_1.$$

Making $\rho \to 1$ we have the desired result

27.65. THEOREM 261.—*Suppose that $\bar{S}(z)$ transform γ into a "star-shaped" domain. If $S(z)$ is subordinate to $\bar{S}(z)$, then* $|s_n|$ [illegible]

Consider

$$G(z) = \int_0^z \frac{\bar{S}(\zeta)}{\zeta} d\zeta = \bar{s}_1 z + \frac{\bar{s}_2}{2} z^2 + \dots.$$

We have $zG'(z) = \bar{S}(z)$, and so, by Theorem 253 and § 27.52 (3),

$$\frac{\partial}{\partial\theta} \arg\left(izG'(z)\right) = \frac{\partial}{\partial\theta} \arg\left(zG'(z)\right) = \frac{\partial}{\partial\theta} \arg \bar{S}(z) = \frac{z\bar{S}'}{\bar{S}} > 0 \quad (z = \rho e^{i\theta}).$$

Let $c(\rho)$ be the transform of $|z| = \rho$ by $G(z)$. Then $\arg\{iz\,G'(z)\}$ is the angle made with the positive real axis by the tangential vector of $c(\rho)$ at the point $w = G(z)$ (for increasing θ). If z moves once round the circle $|z| = \rho$ in the positive sense this angle increases steadily by a total 2π. This evidently implies that $c(\rho)$ is convex; thus $G(z)$ is "schlicht" in γ and transforms it into a convex domain†.

Let now $S(z) = \bar{S}(\omega)$. The function

$$g(z) = G\left(\zeta\omega(z)\right) = g_1 z + g_2 z^2 + \dots \quad (|\zeta| < 1)$$

is subordinate to $G(\zeta z)$, which maps γ on a convex domain. Hence, by Theorem 258,

$$|Q_n(\zeta)| = \left| \sum_{k=1}^{n} a_n^{(k)} \frac{\bar{s}_k}{k} \zeta^k \right| = |g_n| \leqslant |\bar{s}_1 \zeta| \leqslant |\bar{s}_1|.$$

Theorem 259, Corollary 2, now gives

$$|s_n| = \left| \sum_{k=1}^{n} a_n^{(k)} \bar{s}_k \right| = |Q_n'(1)| \leqslant n |\bar{s}_1|.$$

27.7. We turn now to type (C), which is particularly simple for functions S. We define generally for a function f,

$$\varpi(a) = \varpi(a, f) = \Pi\left(\frac{1}{\rho_n(a)}\right),$$

where (as in Theorem 214) $z_1(a)$, $z_2(a)$, ... are the non-zero roots, in order of increasing moduli $\rho_1(a)$, $\rho_2(a)$, ..., of the equation $f - a = 0$, the product being taken over all ρ_n, and a product containing no factors being

† This result provides a new proof of Theorem 252, since, by Theorem 254, applied to the coefficients of G, $|\bar{s}_n/n| \leqslant |\bar{s}_1|$.

interpreted to be unity. $\varpi(a) < \infty$ means, of course, that the product is convergent.

THEOREM 262.—*For a function S,*

$$(1)\qquad \varpi(a) = \varpi(a, S) \leqslant 1+2\frac{d}{|a|}+2\sqrt{\left(\frac{d}{|a|}+\frac{d^2}{|a|^2}\right)} \quad (a \neq 0),$$

$$(2)\qquad \varpi(0) \leqslant \frac{\bar{s}_1}{|s_\nu|} \leqslant \frac{4d}{|s_\nu|},$$

where s_ν is the first coefficient in $\Sigma s_n z^n$ that does not vanish, and where $d = d(\Gamma)$ is defined as in §25.1. (2) gives in particular $|s_\nu| \leqslant \bar{s}_1 \leqslant 4d$.

For a "schlicht" function $\bar{S}$ there is at most one $\bar{z}_n(a)$, $\bar{z}(a)$ say, when $a \neq 0$. By Theorems 214 and 243 we have

$$(3)\qquad \prod_{n\leqslant N}\left(\frac{\rho}{\rho_n(a)}\right) \leqslant \frac{\rho}{\bar{\rho}(a)},$$

where N is determined by $\rho_N(a) < \rho \leqslant \rho_{N+1}(a)$. But the left side of (3) is not increased either if we increase or if we decrease the value of N (for fixed ρ), and it follows that (3) is valid for all values of n and ρ. Making $\rho \to 1$, we have therefore, for any N,

$$\prod_{n\leqslant N}\left(\frac{1}{\rho_n(a)}\right) \leqslant \frac{1}{\bar{\rho}(a)}.$$

Hence

$$(4)\qquad \varpi(a) \leqslant \frac{1}{\bar{\rho}(a)}.$$

Now by Theorems 242 and 243

$$|a| = |\bar{S}\{\bar{z}(a)\}| \leqslant \frac{|\bar{s}_1|\bar{\rho}(a)}{\{1-\bar{\rho}(a)\}^2} \leqslant \frac{4d\bar{\rho}(a)}{\{1-\bar{\rho}(a)\}^2},$$

so that
$$\frac{1}{\bar{\rho}(a)} \leqslant 1+2\frac{d}{|a|}+2\sqrt{\left(\frac{d}{|a|}+\frac{d^2}{|a|^2}\right)},$$

and the first part of the theorem follows from (4).

If $a = 0$, we have $\mu = 1$, $\Pi = 1$, in the right-hand side of the in equality of Theorem 214, and so

$$|s_\nu|\rho^{\nu-1}\prod_{n\leqslant N}\left(\frac{\rho}{\rho_n(0)}\right) \leqslant |\bar{s}_1| \leqslant 4d,$$

first for a special N, and then, as above, for all N and ρ. Making first $\rho \to 1$ and then $N \to \infty$ we obtain the second part.

For special functions $\bar{S}$ $\bar{\rho}(a)$ is known, and (4) gives more precise results. Consider, for example, functions $B(z)$ (§ 23 . 4). Here

$$\bar{B}(z) - b_0 = \frac{2\beta_0 z}{1-z} \quad (b = \beta_0 + i\gamma_0),$$

which is an $\bar{S}$, $\bar{S}_1(z)$ say;

$$\bar{z}(a; \bar{B}) = \bar{z}(a - b_0, \bar{S}_1) = \frac{a-b_0}{a+b_0'} \quad (a \neq b_0),$$

and we have

$$\varpi(a, B) \leqslant \left| \frac{a+b_0'}{a-b_0} \right| \quad (a \neq b_0),$$

where b_0' is the conjugate of b_0. Similar results hold, *e.g.*, for functions C (§ 26 . 3), and for functions f satisfying $|f| \leqslant 1$ in γ.

28. *Various developments.*

28 . 1. THEOREM 263.—*Suppose that $f(z) = a_1 z + a_2 z^2 + \ldots$ is regular in $|z| \leqslant 1$, and that $a_1 = 1$. Let $r(f)$ be the radius of the greatest*† *circumference $|w| = r$ [not circle $|w| \leqslant r$], all of whose points are values taken by f in $|z| \leqslant 1$. Then $r(f) \geqslant A$*‡.

If we add the hypothesis that f is "schlicht" the theorem is contained in Theorem 243; *the additional hypothesis is, however, unnecessary.*

Before proving the theorem we generalize it a little further. Let $\mu(f) = \underset{|z| = \frac{1}{2}}{\text{Max}} |f|$, and let us replace the hypothesis $a_1 = 1$ by $\mu(f) = 1$. To see that the change does generalize the theorem, suppose the final form true. If now $a_1 = 1$, Cauchy's inequality $|a_1| \leqslant r^{-1}M(r)$ shows that $\geqslant \frac{1}{2}$, whence, writing $\phi = f/\mu(f)$, we have $r(\phi) > A_1$,

$$r(f) = r(\phi)\, \mu(f) > \tfrac{1}{2} A_1.$$

We have, then, to prove:

If f is regular in $|z| \leqslant 1$, $f(0) = 0$, and $\mu(f) = 1$, then $r(f) \geqslant A$.

† The values of r, of which $r(f)$ is to be the greatest, form a closed set.

‡ It is not known whether $r(f) \geqslant \frac{1}{4}$ is true; no stronger result is, of course, possible, as the example $\sigma_0(z)$ shows.

This follows easily from Theorem 227. In fact, if $r(f) < k$, then, for appropriate values of α and β, $ke^{i\alpha}$ and $2ke^{i\beta}$ are missing values of f, and

$$\phi = \frac{f-ke^{i\alpha}}{2ke^{i\beta}-ke^{i\alpha}}$$

is a $P(z)$, with $|p_0| \leqslant 1$. Hence

$$\mu(\phi) \leqslant A_1,$$

$$1 = \mu(f) \leqslant k+3kA_1,$$

$$k \geqslant (1+3A_1)^{-1},$$

and this proves the result.

28.2. *A theorem on integral functions*†. What are the conditions that an integral function of an integral function should be of finite order? The answer seems very obvious, but the only known proof has to appeal to the very sophisticated Theorem 263.

THEOREM 264.—*Suppose that f, g, h are integral functions of z, and that $f = g(h)$. If now f is of finite order, then either* (i) *h is a polynomial and g is of finite order*, *or* (ii) *g is of zero order and h is of finite order.*

If we set out to prove that g and h are not both of "large" order we naturally begin a *reductio ad absurdum*: "h is of large order in z for some z, say $|h| = R$; $g(w)$ is of large order in R on $|w| = R$". But this much does not prove that $f(z)$ is of large order; the difficulty is that the $h(z)$ that have $|h| = R$ may have the wrong *amplitude* to make $g(w)$ large in R. The point is met by the following

LEMMA.—*Let*

$$F(r) = M(r,f), \quad G(r) = M(r,g), \quad H(r) = M(r,h).$$

There exists an absolute positive constant a such that, if $h(0) = 0$, then

$$F(r) \geqslant G\{aH(\tfrac{1}{2}r)\}.$$

Let $$\phi(\zeta) = \frac{h(r\zeta)}{H(\frac{1}{2}r)}.$$

is regular in $|\zeta| \leqslant 1$ and $\mu(\phi) = 1$. By Theorem 263 (generalized) there

† No knowledge of the special theory of integral functions is required here. An integral function $f(z)$ is said to be of finite order ρ if $|f| < \exp(|z|^{\rho+\epsilon})$ for (arbitrary positive ϵ and) large $|z|$, and if ρ is the smallest number with this property.

exists an absolute constant a and an $R \geqslant aH(\frac{1}{2}r)$ such that every w satisfying $|w| = R$ is a value of $h(z)$ in $|z| \leqslant r$. There is a w_0 such that

$$|w_0| = R, \quad |g(w_0)| = G(R) = G(|w_0|).$$

There exists a z_0 satisfying $|z_0| \leqslant r$, for which $w_0 = h(z_0)$. Then

$$G\{aH(\tfrac{1}{2}r)\} \leqslant G(|w_0|) = |g(w_0)| = \left|g\big(h(z_0)\big)\right| \leqslant F(r),$$

the result of the lemma.

28.3. Consider now Theorem 264. We may suppose $h(0) = 0$ [otherwise let $h^* = h - h(0)$, $g^*(w) = g\{w + h(0)\}$; then $g(h) = g^*(h^*)$, and the result is true for g, h if it is true for g^*, h^*] and that neither g nor h is a constant. Let

$$h(z) = a_1 z + \ldots + a_n z^n + \ldots, \quad g(w) = \Sigma b_n w^n,$$

and let m be an integer for which $a_m \neq 0$. We have

$$F(r) < K\exp(r^k),$$

$$H(r) \geqslant |a_m| r^m,$$

$$G(a|a_m|2^{-m}r^m) \leqslant G\{aH(\tfrac{1}{2}r)\} \leqslant F(r) < K\exp(r^k),$$

$$G(a|a_m|2^{-m}r) < K\exp(r^{k/m}). \tag{1}$$

We now distinguish two cases :

(a) h is not a polynomial. Then m may be taken arbitrarily large, and it follows from (1) that g is of order zero. Further we have, for every n,

$$|b_n\{aH(\tfrac{1}{2}r)\}^n| \leqslant G\{aH(\tfrac{1}{2}r)\} < K\exp(r^k). \tag{2}$$

Since $|b_n| > 0$ for some $n > 0$ (g not being a constant), (2) shows that h is of finite order.

Case (b). h is a polynomial. Here (1) shows that g is of finite order.

We observe that case (ii) (with h not a polynomial) is a possible one. It occurs, for example, if h is any function of finite order and

$$g(w) = \Sigma e^{-n^2} w^n.$$

A comparison of $\Sigma e^{-n^2} R^n$ with the integral $\int e^{-x^2} R^x dx$ shows without difficulty, in fact, that

$$G(R) < A\exp(A\log^2 R),$$

whence if h is of order ρ, f is of order 2ρ at most (in point of fact of order 2ρ exactly).

28.41. The inequalities (3) for functions C (§ 2.63) show that a set of missing values filling a half-line restricts the "order" of the function to that of $(1-\rho)^{-2}$ at most; Theorem 233 shows that the same restriction is effected by a discrete set $(-4\pi^2 n^2)$ of points on a half line†. Finally Theorem 242 shows that the same restriction of order is effected by any Γ of missing values, for example, any Jordan curve extending to ∞‡. A comparison of these results inevitably suggests that if the curve, in its turn, is replaced by a discrete string of points, with gaps not too large, the order of the function will still not exceed $(1-\rho)^{-2}$. Since it combines the depth of the very special modular theorems with the generality of the "schlicht" functions, this theorem can hardly be easy. Provided, however, that we abandon the ideal of a *best possible* power of $(1-\rho)^{-1}$ for the order, we can not only prove the suggested result, but extend it materially. It will appear, in particular, that a very sparse set of missing values is enough to reduce the order of the function below a constant power of $(1-\rho)^{-1}$.

THEOREM 265.—*Suppose that we are given an integer $k \geqslant 0$, a constant $c > 1$, and an infinite sequence $w_1, w_2, \ldots,$ where $|w_n| = r_n$, $r_1 > 0$, $r_n \leqslant r_{n+1} < cr_n$, and $r_n \to \infty$ as $n \to \infty$. Suppose now that, in γ, $f(z) = a_0 + a_1 z + \ldots$ is regular and takes no value w_n more than k times. Then*

$$M(\rho, f) < A(k)\, m(1-\rho)^{-h},$$

where $\quad h = A(k)\, c(r_1+1), \quad m = \mathrm{Max}\,(1, |a_0|, |a_1|, \ldots, |a_k|).$

The proof depends on

LEMMA 4.—*Suppose that, in γ, $\phi(z) = u_0 + u_1 z + \ldots$ is regular and takes neither of the values 0 and 1 more than k times. Suppose, further, that*

(1) $$|u_n| \leqslant 1 \quad (n \leqslant k).$$

Then

(2) $$M(\rho, \phi) < A(k, \rho),$$

(3) $$|u_{k+1}| < A(k).$$

† This is, we may observe in passing, in striking contrast to the effect of even a *domain* of missing values, provided the domain is *bounded*. Thus the function $\exp\left(\frac{1+z}{1-z}\right)$ has a domain $|w| < 1$ of missing values, but its order is as high as $\exp\left(\frac{A}{1-\rho}\right)$.

‡ A precisely similar group of results holds for the order of the n-th coefficient.

The second part follows from the first by taking $\rho = \frac{1}{2}$ in $u_{k+1} | \rho^{k+1} \leqslant M(\rho)$. The first is a particular case of

THEOREM 266.—*Given $m \geqslant 0$, $n \geqslant 0$, positive numbers $k_0, k_1, \ldots, k_n$, $0 < \vartheta < 1$, $R > 0$; there exists a $H = A(m, n, k_0, \ldots, k_n, \vartheta, R)$ with the following property. Let $f(z) = \sum_0^\infty a_\nu z^\nu$ be regular and have at most m 1-points and n zeros in $|z| < R$. Further let $|a_\nu| \leqslant k_\nu$ $(\nu \leqslant n)$. Then*

$$|f| \leqslant H \quad (|z| \leqslant \vartheta R).$$

28.42. This theorem we deduce from two more lemmas.

LEMMA 5.—*Let N be integral and $N \geqslant 0$, and let $K_0, K_1, \ldots, K_N$ be positive numbers. Then there exists a $J = A(N, K_0, \ldots, K_N)$ with the following property. Let $F(z) = \Sigma \beta_n z^n$ be regular and have exactly N zeros in $|z| \leqslant 1$, and let $|\beta_\nu| \leqslant K_\nu$ $(\nu \leqslant N)$. Then*

$$P = \operatorname*{Min}_{|z|=1} |F| \leqslant J.$$

This is true for $N = 0$ since $P \leqslant |\beta_0|$. Suppose then $N > 0$; also that $P > 0$. Let $\xi_1, \ldots, \xi_N$ be the zeros, and let $G(z)$ be defined by

$$\prod_1^N (z - \xi_\nu) = \prod_1^N (1 - \xi'_\nu z) . F(z)\, G(z), \tag{1}$$

where ξ'_ν is the conjugate of ξ_ν. $G(z) = \Sigma \gamma_\nu z^\nu$ (say) is regular in $|z| \leqslant 1$.

For $|z| = 1$ the two Π's are equal in modulus; hence

$$|G| = 1/|F| \leqslant P^{-1}.$$

Therefore $|\beta_\nu| \leqslant P^{-1}$. The right-hand side of (1) is therefore majorized by

$$(1 + |z|)^N (K_0 + \ldots + K_N |z|^N + \ldots)\, P^{-1} \sum_0^\infty |z|^\nu,$$

a fortiori by

$$2^N \sum_1^\infty |z|^\nu (K_0 + \ldots + K_N |z|^N + \ldots)\, P^{-1} \sum_0^\infty |z|^\nu. \tag{2}$$

Hence $\quad$ | *coefficient of z^N in* (1) | $\leqslant$ *coefficient of* $|z|^N$ *in* (2),

$$1 \leqslant 2^N P^{-1} \{\textit{coefficient of } z^N \textit{ in } (K_0 + \ldots + K_N |z|^N)(1 - |z|)^{-2}\},$$

$$P \leqslant J.$$

28.43. LEMMA 6.—*Let $E > 0$, $0 < R_1 < R_2$. There exists an $\Omega = A(E, R_1, R_2)$ with the following property. Let $f(z)$ be regular*

and never equal to 0 *or* 1 *in* $R_1 < |z| < R_2$, *and let* $R_3 = \frac{1}{2}(R_1+R_2)$ *and*

$$\operatorname*{Min}_{|z|=R_3} |f| \leqslant E.$$

Then $$|f| \leqslant \Omega \quad (|z| = R_3).$$

Let $R_2 - R_1 = 2r$, and let z_0 be a point on $|z| = R_3$ for which $|f(z_0)| \leqslant E$. By Theorem 227†

$$|f| < A_1(E) = E_1 \quad (|z-z_0| \leqslant \tfrac{1}{2}r).$$

Let z_1 be the point in which $|z-z_0| = \frac{1}{2}r$ cuts $|z| = R_3$; then

$$|f| < A_1(E_1) = E_2 \quad (|z-z_1| \leqslant \tfrac{1}{2}r).$$

This process can be continued, and covers the circumference $|z| = R_3$ in $A(r/R_3) = A(R_1, R_2)$ stages. Lemma 6 follows.

28.44. To prove Theorem 266 we now divide the interval ϑR to R into $m+n+1$ equal parts. For at least one of them, which we call (R_1, R_2) we have

$$f \neq 0, 1 \quad (R_1 < |z| < R_2).$$

Also, taking F of Lemma 5 to be $f(R_3 z)$ and observing that $|\beta_\nu| \leqslant k_\nu R_3^\nu \leqslant k_\nu R^\nu$, $N \leqslant n$, we have

$$\operatorname*{Min}_{|z|=R_3} |f| = \operatorname*{Min}_{|z|=1} |F| \leqslant E = A(N, k_0, k_1 R, \ldots, k_N R^N) \leqslant A(n, k_0, \ldots, k_n, R).$$

By Lemma 6

$$|f| \leqslant \Omega = A(E, R_1, R_2) = A(m, n, k_0, \ldots, k_n, \vartheta, R)$$

for $|z| = R_3$. *A fortiori* this holds for $|z| = \vartheta R$.

28.45. We have now established Lemma 4. Return now to Theorem 265; it is enough to prove $|f(\rho)| \leqslant A(k) m (1-\rho)^{-k}$. Let

$$F(z) = f\{\rho + (1-\rho)z\} = b_0 + b_1 z + \ldots,$$

where $$b_n = \frac{(1-\rho)^n}{n!} f^{(n)}(\rho).$$

Since $|\rho + (1-\rho)z| < 1$ in γ, the function $F(z)$ is regular in γ and takes no value w_n more than k times. Let

(1) $$\mu = \mu(\rho) = r_1 + |b_0| + |b_1| + \ldots + |b_k| \geqslant r_1.$$

It is evident from the hypotheses about the w_n that *there exists a* $\nu > 1$ *such that*

$$2\mu \leqslant |w_\nu| \leqslant 2c\mu.$$

† Note that Theorem 227 is Theorem 266 with $m = n = 0$.

Thus

(2) $$|w_\nu - w_1| \leqslant 2|w_\nu| \leqslant 4c\mu$$

and

(3) $$\mu \leqslant 2\mu - r_1 \leqslant |w_\nu| - |w_1| \leqslant |w_\nu - w_1|.$$

Let now $$\phi(z) = \frac{F(z) - w_1}{w_\nu - w_1} = u_0 + u_1 z + \ldots.$$

Clearly ϕ takes neither of the values 0 and 1 more than k times; also, for $0 < n \leqslant k$,

$$|u_n| = \left|\frac{b_n}{w_\nu - w_1}\right| \leqslant \frac{\mu}{|w_\nu - w_1|} \leqslant 1,$$

and $$|u_0| = \frac{|b_0 - w_1|}{|w_\nu - w_1|} \leqslant \frac{|b_0| + r_1}{\mu} \leqslant 1.$$

Thus Lemma 4, (3), is applicable to ϕ, and we have

$$|u_{k+1}| < A(k),$$

(4) $$|b_{k+1}| = |u_{k+1}(w_\nu - w_1)| \leqslant 4c\mu A(k) < cA(k)(r_1 + |b_0| + \ldots + |b_k|)$$
$$< h(1 + |b_0| + |b_1| + \ldots + |b_k|),$$

where we denote by h a constant of the form $A(k)\,c(r_1 + 1)$ (not always the same at different occurrences). (4) may be written

(5) $$(1-\rho)^{k+1}|f^{(k+1)}(\rho)| < h_1\Big((1-\rho)^k|f^{(k)}(\rho)| + \ldots + (1-\rho)|f'(\rho)| + |f(\rho)| + 1\Big)$$
$$= h_1 T,$$

say. Now let $\psi(\rho) = (1-\rho)^{h_1+1}T$. Since $D|F(\rho)| \leqslant |F'(\rho)|$, where $D\Phi(\rho)$ is the upper right-hand derivate of $\Phi(\rho)$, we have

$$D\psi(\rho) = (1-\rho)^{h_1+1}\sum_{n=0}^{k}(1-\rho)^n D|f^{(n)}(\rho)|$$
$$-\sum_{n=0}^{k}(h_1+1+n)(1-\rho)^{h_1+n}|f^{(n)}(\rho)| - (h_1+1)(1-\rho)^{h_1}$$
$$\leqslant (1-\rho)^{h_1}\sum_{n=1}^{k+1}(1-\rho)^n|f^{(n)}(\rho)| - (h_1+1)(1-\rho)^{h_1}\Big(\sum_{n=0}^{k}(1-\rho)^n|f^{(n)}(\rho)| + 1\Big)$$
$$\leqslant (1-\rho)^{h_1}\Big\{(1-\rho)^{k+1}|f^{(k+1)}(\rho)| - h_1\Big(\sum_{n=0}^{k}(1-\rho)^n|f^{(n)}(\rho)| + 1\Big)\Big\}$$
$$\leqslant 0,$$

by (5). Hence

$$\psi(\rho) \leqslant \psi(0) = \sum_{n=0}^{k} |f^{(n)}(0)| + 1 < A(k)(1+|a_0|+\ldots+|a_k|) < A(k)m,$$

and so finally $$(1-\rho)^{k_1+1}|f(\rho)| < \psi(\rho) < A(k)\,m.$$

28.5. It is natural to generalize "schlicht functions" to "functions of valency p", which take (in γ) no value more than p times. The questions at once arise: are such functions of order $2p$ and coefficient order n^{2p-1} (at most)? The answers are affirmative. The proofs, due to Cartwright†, are difficult, and depend on ideas unlike any we have been considering. A further important generalization is to functions of "mean valency p", which, in a certain defined sense, take values "on the average" not more than p times: p now need not be integral (and may be less than 1). [See D. C. Spencer, *Trans. American Math. Soc.*, 48 (3) (1940), 418–435, and references there given.] Cartwright's theorems (and to a great extent their proofs) are, rather surprisingly, true for the wider class, and this greatly extends their scope.

* M. L. Cartwright, *Math. Annalen*, 111 (1935), 98–118.

Addenda and Corrigenda.

P. 25. Before § 3 insert the following:

Suppose that $f_n(z) \to f(z)$ at each point z, or (x, y)†, of a bounded closed set R. Let $d_n(z) = f_n - f$, so that $d_n \to 0$ at each z of R.

The convergence of f_n to f (or of d_n to 0) is uniform in R if and only if $d_n(z_n) \to 0$ for every *function (or sequence) z_n of n (z_n always belonging to R).*

Let M_n be the upper bound of $|d_n|$ in R. Uniform convergence is equivalent to $M_n \to 0$. Also $|d_n(z_n)| \leqslant M_n$ for *all* functions z_n, $|d_n(z_n)| \geqslant \frac{1}{2}M_n$ for *some* function z_n.

Suppose $f_n \to f$ at each z of a bounded closed R, and that each f_n is continuous in R. Then (i) *if the convergence is uniform the continuity (of f_n) is uniform (in n)*‡; (ii) *conversely, if the continuity is uniform so is the convergence.*

Further, (iii) *if f_n is uniformly continuous in R, and convergent, to f say, in a set E dense in R, then f_n converges uniformly to a limit f in R.*

(i) Given the convergence uniform, f is continuous, $f_n - f$ is continuous, and we may suppose $f = 0$. If all z's concerned belong to R, and $\Delta z = z' - z$, $\Delta F = F(z') - F(z)$, we have

$$|\Delta f_n| \leqslant |f_n(z')| + |f_n(z)| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon \quad (n > n_0),$$

where $n_0 = n_0(\epsilon)$ is independent of z and z'. Also $f_1, f_2, \ldots, f_{n_0}$ are continuous. It follows that

$$|\Delta f_n| < \epsilon \quad \text{for} \quad |\Delta z| < \delta(\epsilon), \quad \text{independent of } n.$$

(ii) Given the continuity uniform, we have

$$|\Delta f_n| < \epsilon \quad \text{for} \quad |\Delta z| < \delta(\epsilon).$$

Hence $|\Delta f| = \lim |\Delta f_n| \leqslant \epsilon$ for $|\Delta z| < \delta$; f is continuous, $f_n - f$ is uniformly continuous, and we may suppose $f = 0$. If the convergence is not uniform, there exists a function z_n giving $|f_n(z_n)| > a > 0$. If ζ is

† The results that follow are true in n dimensions; $n = 2$ is perfectly typical.

‡ The continuity of an F continuous in (a closed) R being necessarily uniform in z of R, reference to it is usually suppressed; any uniformity actually mentioned is with respect to some further parameter. The f_n of the text is thus continuous uniformly in z and n.

a limit point of the z_n, $f_n(\zeta)$ differs little from $f_n(z_n)$ when z_n is near ζ, by the uniform continuity. Hence $|f_n(\zeta)| > \frac{1}{2}a$ for some large n, contrary to $f_n(\zeta) \to 0$.

(iii) It is enough, by (ii), to prove f_n convergent at each z of R. Now given ϵ there exists a $\delta(\epsilon)$ (independent of n) such that for all n $|\Delta f_n| < \frac{1}{3}\epsilon$ whenever $|\Delta z| < \delta$. If now z is any point of R, there exists a point $z' = z'(z, \epsilon)$ of E for which $|\Delta z| < \delta$. Then

$$\begin{aligned} |f_n(z) - f_m(z)| &\leqslant |f(z') - f_m(z')| + |\Delta f_n| + |\Delta f_m| \\ &\leqslant |f_n(z') - f_m(z')| + \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon. \end{aligned}$$

The first term on the right, however, is less than $\frac{1}{3}\epsilon$ for

$$m, n > N(z', \epsilon) = N(z, \epsilon),$$

on account of the convergence of f_n at z' (a point of E). We have, then,

$$|f_n(z) - f_m(z)| < \epsilon \quad [m, n > N(z, \epsilon)],$$

and f_n is convergent at z.

We can use these results with advantage later. Meanwhile, the "Continuous Selection Principle", namely the Corollary of Theorem 5, is an immediate consequence.

Note first, however, a correction and an addition to the Corollary. (i) At the end of the first sentence of the enunciation add: "*and f_n is bounded (as $n \to \infty$) for some fixed z of D.*" (ii) Add at the end:

"*There is a corresponding result with a bounded closed set R in place of the open D. If f_n is uniformly continuous in R and bounded at some fixed point of R, then there exists a subsequence (n_r) such that f_{n_r} converges to a continuous f, uniformly in R.*"

After the proof of Theorem 5 we continue: A function F_n, uniformly continuous in a bounded closed set, and bounded at a fixed point of the set, is uniformly bounded in the set. (There is a finite network of squares covering the set, independent of n, and such that $|\Delta F_n| < 1$ in any one square). Hence f_n is uniformly bounded in any D'_- and in R in the respective cases. Choose a denumerable set E dense in D (in R, in the easier "closed" case); by the Theorem there exists a subsequence (n_r) for which f_{n_r} converges to some f in E. Then in any D'_- f_{n_r} converges uniformly to some f, and clearly this involves the existence of an f, continuous in D, such that $f_{n_r} \to f$ uniformly in any D'_-.

P. 37. **Omit Theorem 16** (used only to prove the vital Theorem 21, but originally an unpleasant necessity. Theorem 21 now receives a new proof).

P. 42, l. 6 to l. 15. Substitute: "By the Addendum to p. 25 f_n converges uniformly in E_0 to a continuous limit, which must be f_c."

P. 42, Theorem 21. The first two sentences of the proof stand. For the rest substitute:

We prove first that for any bounded function $h(\theta)$

$$(1) \qquad \int_{-\pi}^{\pi} g_n h \, d\theta \to \int_{-\pi}^{\pi} gh \, d\theta.$$

Suppose $|h| \leqslant C$. For arbitrarily small positive ϵ, δ there exists, by Theorem 6, a step function h^* such that $|h^*| \leqslant C$ and $|h-h^*| < \epsilon$ except in a set X of measure $mX < \delta$. Then

$$\left| \int (g_n h - gh) \, d\theta - \int (g_n h^* - gh^*) \, d\theta \right|$$
$$\leqslant \int_{CX} (|g_n| + |g|) \, \epsilon \, d\theta + \int_X (|g_n| + |g|)(|h| + |h^*|) \, d\theta.$$

The first term on the right-hand side is small with ϵ [since g is integrable L and $M_1(g_n) \leqslant M_r(g_n) \leqslant G$]. The second is small with mX since $|h|+|h^*| \leqslant 2C$, g is integrable L, and $\int |g_n|$ is u.a.c. Hence the right-hand side is small with δ and ϵ. Finally the second integral on the left-hand side tends to 0, since h^* is a step-function. This proves (1).

Now let $\phi = [g]_N$, $\psi = |\phi|^{r-1} \overline{\operatorname{sgn}} \, \phi$. Since ψ is bounded,

$$\int g_n \psi \, d\theta \to \int g\psi \, d\theta$$
$$= \int |g| \, |\phi|^{r-1} d\theta;$$

$$\frac{1}{2\pi} \int |\phi|^r \, d\theta \leqslant \frac{1}{2\pi} \int |g| \, |\phi|^{r-1} \, d\theta$$
$$= \lim \frac{1}{2\pi} \int g_n \psi \, d\theta$$
$$\leqslant \varliminf M_r(g_n) \, M_{r'}(\psi) = \lambda \left(\frac{1}{2\pi} \int |\phi|^r \, d\theta \right)^{1/r'}$$

where $\lambda = \underline{\lim}\, M_r(g_n)$. Since $\int |\phi|^r d\theta$ is finite this gives

$$\frac{1}{2\pi}\int |\phi|^r d\theta \leqslant \lambda^r,$$

(2) $$\frac{1}{2\pi}\int |g|^r d\theta = \lim_{N\to\infty} \frac{1}{2\pi}\int |\phi|^r d\theta \leqslant \lambda^r,$$

which includes the desired result.

P. 43, proof of Theorem 22. This should read: "By Theorem 5, Corollary, there exists a subsequence (n_r) such that $f_{n_r} \to f$ uniformly in (the whole of) E_0. By Theorem 20, $f = \int_0^\theta g\, d\theta + c$". Conclude with the last sentence of the text.

P. 46, Theorem 26. For the proof substitute: "This follows (with f_n for g_n) from (2) of the Addendum to Theorem 21 (p. 42)".

P. 64, § 5.91, Lemma. Replace from "hence" (l. 1 of the proof) to the end of § 5.91 by "and if $|h| \leqslant K$, $|\sigma_n| \leqslant K$ (Theorem 34), and so $|\sigma_n - h| \leqslant 2K$. The result follows by Theorem 10."

P. 66, § 6.1. Add (at appropriate places): "$F(D)$ is a closed set. . . . C is the frontier of each of the domains (and is a closed set). The domains have no common point, and a Jordan curve joining a point of one to a point of the other must have a point in common with C".

4 lines from end. For "continuous" read "bi-continuous".

3 lines from end. For the second "D" read "Δ".

P. 67, line 13 on. We must distinguish L_1, $-L_2$, l_1, $-l_2$, upper and lower bounds for positive t and negative t giving $\rho(t) = r$, and make the obvious modifications in the subsequent argument.

P. 83. From "and" (l. 18) to "$\geqslant \rho$)." (l. 19) substitute "and let B be a frontier point of E'". After "Clearly" (l. 19) insert "B is interior to D,".

P. 105, end of § 9.2. The "minimum" idea can be applied to prove the existence of a solution of $z^n = k$ also; we therefore give a complete proof of the "fundamental theorem", independent of the theory of circular functions (or processes of integration).

If $f(\zeta) = \zeta^n + a_1\zeta^{n-1} + \ldots + a_n$ is never zero, then, since $|f|$ is large for large $|\zeta|$, $|f|$ attains a minimum other than 0, at ζ_0 say. Let $\zeta = \zeta_0 + z$, $f(z) = A_0 + A_m z^m + \ldots + z^n$, where $A_0 \neq 0$, and A_m is the first of $A_1, A_2, \ldots, A_n = 1$ that is not 0. If $m = 1$ we can choose $z = -\delta A_0/A_1$, when $|f(\zeta)| = |A_0|\{1 - \delta + O(\delta^2)\} < |A_0|$ for small δ, a contradiction. A similar argument succeeds, with $z = \delta u$, and u a solution of $u^m = -A_0/A_m$, provided we can always solve

$$F(\zeta) = \zeta^m - k = 0.$$

Repeat the "minimum at ζ_0" argument on $|F(\zeta)|$ itself. If ζ_0 is not 0 then "A_1" is not 0, and the crude form of the argument succeeds. It remains only to show that for $k \neq 0$ $|F|$ is not a minimum at $\zeta = 0$. We now observe first that we can always solve $\zeta^m = \pm 1, \pm i$. For we can solve $\zeta^2 = a + ib$, $\pm\zeta$ being solutions, with

$$\zeta = \sqrt{\{\tfrac{1}{2}a + \tfrac{1}{2}\sqrt{(a^2+b^2)}\}} + i \operatorname{sgn} b \sqrt{\{-\tfrac{1}{2}a + \tfrac{1}{2}\sqrt{(a^2+b^2)}\}};$$

and we may therefore suppose m odd, in which case ζ^m takes the 4 values $\pm 1, \pm i$ in some order when ζ does. Finally, if u_ν $(\nu = 1, 2, 3, 4)$ are respectively solutions of $u^m = \pm 1, \pm i$, the four points $(\delta u_\nu)^m$ are at the small distance δ^m respectively E, W, N, S of the origin. *One* of these must be nearer the point k than the origin, and $|F|$ is not a minimum at $\zeta = 0$.

Alternatively we can apply the "N, S, E, W" idea at the ζ_0 of the general case $f(\zeta)$.

The history is interesting. The theorem was stated by d'Alembert in 1746, and is called "d'Alembert's theorem" by Gauss. The first "proof", by modern standards, was given by Gauss in 1799; of this Klein says "vom heutigen Standpunkte . . . er ist im Prinzip richtig, aber nicht vollständig". His second proof, not merely complete, but the best from the purely algebraic standpoint even to-day, is dated December 1815. (There is a third of 1816.) In 1815 Argand [*Gergonne's Annales*, 5 (1815), 204] gave a proof by the "minimum" argument, but took the solution of $z^m = k$ for granted (by the "Argand diagram" and circular functions). The ideas of the proof given above are all to be found (heavily overlaid with detail) in Cauchy [*Journal de l'Ecole Polytech.*, 11 (1820), 411; also later in *Exercices de Mathématiques*, 4 (1929), 98, or *Cours de l'Analyse*, Ch. X, 331]. The alternative proof indicated above is given (still overlaid) in Todhunter's "Theory of Equations" (pp. 16–20 in the 3rd edition, 1875; Cauchy's name is mentioned). Since then the proof seems to have been largely forgotten.

P. 110, l. 15. The regularity of ϕ is a consequence of the following very general theorem.

Suppose that for each fixed t of a finite range (a, b) *of the real variable t* $F(z, t)$ *is regular in z of a fixed domain D, and that F is continuous in* (z, t) *for all z of any* D'_- *and t of* (a, b). *Then* $\int_a^b F(z, t)\,dt$ *is a regular function of z in D. The result is valid also for an infinite range* (a, b) *of t, provided* $\int_a^b$ *is convergent uniformly in z of any* D'_-.

This follows immediately from the following "Converse of Cauchy's Theorem" (Morera's Theorem), which we suppose known.

If $F(z)$ *is continuous and one-valued in a domain D, and if for every simple polygon C whose interior is contained in D we have* $\int_C F(z)\,dz = 0$, *then F is regular in D.*

For $$\int_C \left(\int_a^b F(z, t)\,dt\right) dz = \int_a^b \left(\int_C F(z, t)\,dz\right) dt = \int_a^b 0\,dt = 0.$$

[The scope of the "$F(z, t)$" theorem and its dependence on Morera's Theorem have not everywhere been recognized. Morera's Theorem gives a similar immediate proof of Weierstrass's Theorem on the regularity of a uniformly convergent series of regular functions (and its differentiability term by term.)]

P. 112, after § 9.8. We add one or two additional theorems.

THEOREM. *Suppose that* $\lambda > 0$, *and that* $\phi_1(z), \phi_2(z), \ldots, \phi_N(z)$ *are regular in a bounded domain D, and continuous in* D'. *Then*

$$S(z) = \sum_{n=1}^{N} |\phi_n(z)|^\lambda$$

attains its upper bound for D' *at a point of* $F(D)$.

We shall have established this if we show that, given any z_0 of D, either (i) there is a z_1 of D' with $S(z_1) > S(z_0)$, or (ii) there is a z_1 of $F(D)$ with $S(z_1) \geqslant S(z_0)$. [The bound is attained, at ζ say. If ζ belongs to D (i) is impossible for $z_0 = \zeta$.]

Let μ be a typical suffix for which $\phi_\mu(z_0) = 0$, ν one for which Then the $\phi_\nu^\lambda(z)$ are regular at z_0 (whatever branches are

taken). Let

$$F(z) = \sum_{\nu} \phi_{\nu}^{\lambda}(z)\, \overline{\mathrm{sgn}}\, \phi^{\lambda}(z_0),$$

so that $F(z_0) = |F(z_0)| = S(z_0)$.

If $F(z)$ is not constant there is a z_1 near z_0, where $|F(z_1)| > |F(z_0)|$, and so

$$S(z_1) \geqslant |F(z_1)| > |F(z_0)| = S(z_0),$$

a case of (i). If, however, F is constant, it is constant in the whole of D', and for any point z_1 of $F(D)$

$$S(z_1) \geqslant |F(z_1)| = |F(z_0)| = S(z_0).$$

There is an application of this to prove:

Let $\phi(z)$ be regular in $|z| < R$ and continuous in $|z| \leqslant R$. Then $M_{\lambda}(r, \phi)$ is a monotonic increasing function of r in $0 \leqslant r \leqslant R$.

Let $$\phi_n = \phi(ze^{2n\pi i/N}) \quad (1 \leqslant n \leqslant N).$$

Let $r < R$. For any z_0 of $|z| = r$ there exists, by the Theorem, some z_1 of $|z| = R$, varying with N when that varies, such that

$$(1) \qquad \frac{1}{N}\sum_{n=1}^{N} |\phi(z_0\, e^{2n\pi i/N})|^{\lambda} \leqslant \frac{1}{N}\sum_{n=1}^{N} |\phi(z_1\, e^{2n\pi i/N})|^{\lambda}.$$

Since $|\phi(z)|^{\lambda}$ is continuous in θ for $|z| = \rho \leqslant R$, we have, for any ζ_N varying with N but satisfying $|\zeta_N| = \rho$,

$$\lim_{N \to \infty} \frac{1}{N}\sum_{n=1}^{N} |\phi(\zeta_N\, e^{2n\pi i/N})|^{\lambda} = \frac{1}{2\pi}\int_0^{2\pi} |\phi(\rho e^{i\theta})|^{\lambda}\, d\theta.$$

Applying this to (1) with $\rho = r, R$; $\zeta_N = z_0, z_1$, we obtain

$$M_{\lambda}^{\lambda}(r, \phi) \leqslant M_{\lambda}^{\lambda}(R, \phi),$$

and this is equivalent to the desired result.

THEOREM. *Suppose that $f(z)$ is regular for all (finite) z, and suppose for simplicity that $f(0) \neq 0$. Let $M(r) = M(r, f)$, and let $n(r)$ be the number of zeros of f in $|z| \leqslant r$. Then*

$$n(r) < \frac{1}{\log 2} \log M(3r) + K,$$

where K is a constant.

Let a_n $[n = 1, 2, \ldots, N = n(r)]$ be the zeros of f in $|z| \leqslant r$, and let

$$\phi(z) = f(z) \Big/ \prod_{n \leqslant N} \left(1 - \frac{z}{a_n}\right).$$

Then

$$|f(0)| = |\phi(0)| \leqslant M(3r, \phi) \leqslant M(3r, f) \Big/ \Pi\left(\frac{3R}{|a_n|} - 1\right) \leqslant M(3r)\, 2^{-N}.$$

P. 113, l. 4. f/z is bounded as $z \to 0$ since $f'(0)$ exists.

P. 116, end of § 10 . 4. Add: As a typical application we have:

If $f(z)$ is regular for every (finite) z and

$$e^{f(z)} = O(e^{|z|^k})$$

as $z \to \infty$, then $f(z)$ is a polynomial of degree not exceeding k.

Let $f(z) = \Sigma a_n z^n$. For large positive R

$$\mathfrak{R} f(R\zeta) < 2R^k \quad (|\zeta| < 1).$$

By Theorem 110 $$|a_n R^n| \leqslant 2(2R^k - \mathfrak{R} a_0) \quad (n > 0),$$

and making $R \to \infty$ we have $a_n = 0$ if $n > k$.

This result is fundamental in the theory of integral functions, and something like Theorem 109 or 110 is indispensable to its proof.

P. 118, l. 5. *For* "Also . . . . Thus [8 lines lower]" *read* "Since the continuity is uniform in any D'_-".

L. 16. *For* "may . . . deduced" *read* "follow".

P. 119, Theorem 115. Add at the end of the enunciation: "*Further, if r' is sufficiently small, f takes no value more than n times in $|z-a| < r'$*".

Add at the end of the proof [p. 120, l. 3]: "For the last part we observe that if s, r have the foregoing properties, then an $r' \leqslant r$ such that $|f-b| \leqslant s$ for all z satisfying $|z-a| \leqslant r'$ will have the desired property".

P. 120, l. 4 Omit "$f'(z) \neq 0$ in D, and"; and omit the second sentence. Add: "By Theorem 115 f' is never 0 if f is 'schlicht'".

P. 121. At the end add: "Similar arguments give the following result for a circle, in its way best possible.

Let $f(z) = a_1 z + a_2 z^2 + \ldots$, where $a_1 \neq 0$, be regular in $|z| \leqslant r$, and never zero there except at $z = 0$. Let $\underset{|z|}{\text{Min}}\, |f(z)| = m$. Then the inverse function $\phi(w) = b_1 w + b_2 w^2 + \ldots$ (certainly regular and "schlicht" in some circle about $w = 0$) is regular and "schlicht" in $|w| < m$, and satisfies $|\phi| < r$ there.

Suppose the value w satisfies $|w| < m$. Then as z describes $|z| = r$

$$\Delta \arg \{f(z)-w\} = \Delta \arg f(z) + \Delta \arg \left(1 - \frac{w}{f(z)}\right)$$

$$= 2\pi + 0,$$

since $|w/f| < 1$ on $|z| = r$. Thus $f-w$ has exactly one (and a simple) root, $\phi(w)$ say, in $|z| < r$. Let $w+\delta w$ be a value near w; f takes this value somewhere near z, and nowhere else: thus $\phi(w+\delta w) = z + \delta z$, where δz is small. Hence (since $\delta z \to 0$ when δw is made to tend to 0)

$$\lim_{\delta w \to 0} \frac{\delta \phi}{\delta w} = \lim_{\delta w \to 0} \frac{\delta z}{\delta w} = \lim_{\delta z \to 0} \frac{\delta z}{\delta w} = \lim_{\delta z \to 0} 1 \Big/ \frac{\delta w}{\delta z} = 1/f'(z)$$

(where the last denominator is not 0 since z is a simple root of $f-w$). Thus ϕ is one-valued and differentiable in $|w| < m$, and so regular. In this domain it is further the inverse function of f, since $\phi(w)$ is a z giving $w = f(z)$.

P. 132, Lemma 4. Whatever the relations of D to the point at ∞, if z_1 and z_2 are points of $F(D)$ a closed contour composed of points of D cannot *separate* z_1 and z_2. In the proof of the Lemma we can, without reducing to $z_2 = \infty$, suppose z_1, z_2 finite points of $F(D)$ and take $\zeta = \surd\{(z-z_1)/(z-z_2)\}$.

L. 2 of the proof. After "Then" insert "the point at".

P. 141, Lemma 9. After the first two sentences of (a), substitute the following for the rest of the proof of (a): "Since the nuclei D, Δ are *domains*, they contain $z = 0$ and $\zeta = 0$ as *interior* points; also

$$f(0) = \lim f_n(0) = \lim 0 = 0,$$

and similarly $\phi(0) = 0$. If now f is constant, its value is 0, and since $f_n \to f$ uniformly in any D'_-, we have $f_n(z_n) \to 0$ for an arbitrary z_n of any D'_-. Let η be a small positive constant, so that $|\zeta| \leqslant \eta$ is a Δ'_-, and let ζ_0 be a point other than 0; for large n $\Delta_n \supset \Delta'_-$ and ϕ_n is defined at ζ_0; let $z_n = \phi_n(\zeta_0)$. ϕ is continuous, so $\phi(\zeta_0)$ is small; so also then is $\phi_n(\zeta_0)$, since $\phi_n \to \phi$ at ζ_0. Hence $z_n = \phi_n(\zeta_0)$ is small, and belongs to a D'_-, and also to D_n. Consequently $f_n(z_n) \to 0$, a contradiction since the left-hand side is ζ_0".

(*b*) and the first two sentences of (*c*) stand. Continue: "It is enough to show that any $\delta'_- \subset \Delta_n$ for large n. If D'_- corresponds to δ'_- by f, and $D'_- \subset D'_0 \subset D$, we have $D'_0 \subset D_n$, the image of D'_0 by f_n contains δ'_-, *a fortiori* Δ_n contains δ'_-".

[(*d*) stands).]

P. 142, § 17.3. In l. 8 insert "in Δ" after "Φ". In l. 10 insert "By Vitali's Theorem, $\Phi_m \to \Phi$ in Δ^*" after the full stop. Substitute for the passage from "It follows . . ." in l. 11 to the end of Case (i) the following: "Thus Δ^* and so Φ are completely determined, and every convergent subsequence leads to them. Hence $\phi_n \to \Phi$, and Δ^*, Φ are respectively Δ and the inverse of f".

P. 142, last line. Delete from "and" to "δ", and delete "D_0". For the first three lines of p. 143 substitute: "z-domain (D_- contains $z = 0$). Since $D'_- \subset D_n$ for large n, we have $\delta'_- \subset \Delta_n$; δ_- being arbitrary this gives $\delta \subset \Delta$, which is false".

Minor or small corrigenda.

P. 20, f.n. In the second inequality the k's should be κ's.

P. 33. Delete the Corollary, the line above, and the proof.

P. 34, Theorem 13. Delete the line after (5).

P. 35. Delete from "To see . . ." (l. 11) to "$= 2$" (l. 15).

P. 39, Theorem 19. The necessity part is a luxury from the point of view of applications.

P. 41, ll. 5–7. Substitute for the expression in l. 7 "$\overline{\lim}_{\theta \to \theta_0} f(\theta)$".

Note : the same points occur several times later; we shall sometimes indicate the alteration by the phrase "use $\overline{\lim}$".

P. 48, l. 4 from bottom. *For* f_n *read* f_{n_r}.

P. 71, Theorem 49. Use "$\overline{\lim}_{\to P \text{ in } D}$".

P. 78, last four lines. "$U_+(\psi_0) = \overline{\lim}_{\psi \to \psi_0} U(\psi)$, $U_-(\psi) = \underline{\lim}_{\psi \to \psi_0} U(\psi)$."

P. 81, Theorem 63. In the second and third lines of the proof, *for* D *read* D', and insert "continuous" before "limit".

P. 90. A less awkward notation would be u_+, u_- for u_1, u_2 (leaving the rest, *e.g.* $U_{1,2}$, unaltered). Note a slip: in l. 9 below (2), *for* u_1 *read* u_2.

P. 103, Theorem 101. Use " $\overline{\lim}_{z \to \xi \text{ in } D} |f| \leqslant M$ ".

P. 104, Corollary 3, l. 2. Before "constant" insert "the same".

P. 105, Theorem 102; p. 106, Theorem 102*a*; p. 107, Theorem 103: Use $\lim_{\to \xi \text{ in } D} |f| \leqslant M$".

P. 107, l. 3 from bottom. After "H" insert "depending on ϵ (at any rate in the possible case of $M = 0$)".

P. 108. In lines 1, 2 for "M" substitute "$M + \epsilon$". Add at the end of l. 3: "and then $\epsilon \to 0$". Compare the f.n., p. 109.

P. 110, Theorem 106. Use "$\overline{\lim}\, \psi \leqslant M$".

P. 113, l. 3. ϕ is bounded near $z = 0$ since $f'(0)$ exists.

P. 114, l. 13. *For* "A" *read* "U".

P. 117, l. 2. *For* $|\Re f|$ *read* $\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Re f|\, d\theta$.

P. 121, l. 5 from bottom. *For lying in D read containing only points of D.*

P. 122, l. 12. After "every" insert "convergent".

P. 123, § 11.52, (2). After the parenthesis insert "We show that if ξ is any point of Λ then $f \neq \xi$ in D."

P. 133, l. 12. *For DP read P.*

P. 138, ll. 6, 7 from bottom. *For* since $\ldots = M(\Phi)$ *read* by Theorem 120, Cor. 1.

P. 142, l. 4. Should begin "(*d*) $\Delta \subset \delta$".

In Theorems 62, 63, 64, 102, 102*a*, 106, 113, 116, and in Lemma 8 (p. 134), the domain D should be given bounded.

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